Contents

1 Introduction

Second price auctions, also called Vikrey auctions after their founder, are very usefull to deal with ressource sharing. In fact they control the division of ressources among players and the price they will have to pay for the ressouces they get according to their bids. The principle is to collect the bids from every player and to allocate ressources to those whose bids are among the highest . Then they have to pay what those who did not get any ressources would have paid for that. It is called "Exclusion Compensative Principle of Vikrey" since it excludes some players and then compensates the prices.

In this document we will present and compare two methods of sharing a quantity Q of ressources. They both implement the principle of Vikrey but in very different ways. The first method has been found by N. Semret and is iterative since it requires many exchanges between any single player and the owner of the ressources. The second method is instantaneous and has been found by our team, in fact it only requires one exchange between a player and the owner to set the attribution for that player. In the first part of this document we will present the method used by N. Semret and the main results he obtained. Then we will show the main weaknesses of this method. In a third part we will present the method we founded and the results he obtained with it.

2 Progressive Second Prices - PSP

2.1 Nash Equilibria

First and foremost, we need to define what are Nash Equilibria and ε-Nash equilibria . To sum up a Nash Equilibrium is a state of the auctions were, if the other players do not move, no player has an incentive to change his bid. Due to the fact that he would diminish his gain. The property of a Nash Equilibrium is that players can predict it, and predict that their opponents will also predict it. The auction is in a ε-Nash equilibrium state when no players may earn more than ε with a new bid if he moves unilaterally.

2.2 Bids

Along the auction the players emmit bids. With these bids are computed allocations for each player. A bid is a set of two values, the first q_i is the share of ressource the player anounces to want to get from the owner, the second p_i is the unitary price he is willing to pay for it. Hence any single player bid looks like s_i defined in (1);

$$s_i = (q_i, p_i) \in S_i = [0, Q] \times \mathbb{R}^+ \tag{1}$$

2.3 Pinciple

As said in the introduction this method uses the "Exclusion Compensative Principle of Vikrey". N. Semret first thought that his method was ending up on a Nash Equilibrium . But he then considered ε as a bid fee that players have to pay to emit another bid. He proved that the auction was instead reaching a ε-Nash equilibrium in a finite number of bids. The principle is thus to collect players bids and to compute allocations for each of them. The allocations are then sent back to the players who emit others bid correlated to their allocation and the bids of his opponents. The auction ends when no player has an incentive to emit a new bid.

2.4 PSP Rule

As previously said, the author collects each of the n player's bid so to have $s = (s_1, ..., s_n)$. It then computes the allocations (see (2),(3) and (4)) for each player according to the rules.

$$A_i(s) = (a_i(s), c_i(s)) \in [0, Q] \times \mathbb{R}^+ \tag{2}$$

$$\begin{aligned} a_i(s) &= q_i \wedge [Q - \sum_{j \neq i, p_j \geq p_i} q_j]^+ & (3) \\ c_i(s) &= \sum_{j \neq i} p_j \cdot [a_j(0, s_{-i}) - a_j(s_i, s_{-i})] & (4) \end{aligned}$$

Being given his allocations and the former bids of other players, the considered player then computes his next bid (see (5), (6), (7) and (8)) providing he is not earning fewer than ε in emitting the new bid. Otherwise the players keeps the same bid.

$$\begin{aligned} P_i(z, s_{-i})) &= \inf\{y \geq 0 : [Q - \sum_{j \neq i, p_j \geq y} q_j]^+ \geq z\} & (5) \\ G_i(s_{-i}) &= \{z \in [0, Q] \ / \ \theta_i'(z) \geq P_i(z, s_{-i})\} & (6) \\ v_i &= [supG_i(s_{-i}) - \frac{\varepsilon}{\theta_i'(0)}]^+ & (7) \\ t_i &= (v_i, \theta_i'(v_i)) & (8) \end{aligned}$$

$P_i(z, s_{-i}))$ being the minimal price to bid to get a quantity z
G_i being the quantity space under which i has a positive valuation.
$E_i = (v_i, t_i) \in [0, Q] \times \mathbb{R}^+$ being the ε-best reply

To sum up the PSP-Algorithm can be seen as follows:
PSP-ALGORITHM

1. *$s_i = 0$ and $s_{-i} = \emptyset$. Beginning of a independant thread which collects the modifications of s_{-i};*
2. *Computation of the ε-best reply defined by the strategy t_i;*
3. *If $u_i(t_i, s_{-i}) > u_i(s_i, s_{-i}) + \varepsilon$, send the bid $s_i = t_i$;*
4. *sleep one second;*
5. *go back to **2**.*

Using the definitions below:
u_i being the utility of player i : $u_i(s) = \int_0^{a_i(s)} [\theta_i^{'}(z) - P_i(z, s_{-i})] dz$
$\theta_i^{'}$ being the marginal evaluation of the player i

3 Weaknesses of PSP-model

3.1 Allocation problems

Before having published the PSP-model as it is known today, N. Semret had published a first unfeasible model of his theory (see (9)). The advantages of this model were that it could reach a Nash Equilibrium and no more a ε-Nash equilibrium , it could also better deal with equality in the bids of players. There was for any single player a best reply (see (10) for definition) and not a ε-best reply .

$$a_i(s) = q_i \wedge [Q - \sum_{j \neq i, p_j > p_i} q_j(s)]^+ \tag{9}$$

$$\forall i \in \Im, \forall s \in S, \;\; u_i(q_i, p_i; s_{-i}) \leq u_i(q_i, \theta_i^{'}(q_i); s_{-i}) \tag{10}$$

The problem of this model is that in some cases it could give an unfeasible allocation. For example given the case :

$$\begin{array}{rcl} \text{Q} & = & 10 \\ \text{n} & = & 3 \\ s_1 & = & (3,10) \\ s_2 & = & (7,5) \\ s_3 & = & (5,5) \end{array}$$

The allocation obtained with the former PSP-model gives these quantities[1]:

[1] We then obtain $a_1 + a_2 + a_3 > Q$ which is not feasible

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 7 \\ a_3 &= 5 \end{aligned}$$

3.2 Equalities management

Because of the problems caused by the first model N. Semret decided to design a new model described in (3). However this new model does not lead to a Nash Equilibrium but to a ε-Nash equilibrium and it has a real problem to manage equalities. In fact when two players bid the same price they are both penalized. For example, given the case :

$$\begin{aligned} \mathrm{Q} &= 100 \\ \mathrm{n} &= 2 \\ s_1 &= (70{,}15) \\ s_2 &= (60{,}15) \end{aligned}$$

The allocation is as follows :

$$\begin{aligned} a_1 &= (100-60) &= 40 \\ a_2 &= (100-70) &= 30 \end{aligned}$$

Which means that a quantity of 30 is lost with this method. We notice however that the one who bid for the biggest quantity is less penalized than the other. This problem introduces another one which is far more hazardous for the auction. In fact a collusion of three players may cheat on their bids to make one of them be granted a great quantity of ressources and pay a price zero. The case which leads to that problem can be described as follows:

1. Every player granted a quantity is fully served
2. The two first players who are not served have bid the same price
3. These two players have both a greater quantity than the one of the players served

In that very case every served player gets his quantity for a price zero, which is very hazardous for the owner of the ressources.

3.3 Cheating behavior

Apart from the cheat which has been described uphere, there is another treacherous behavior allowed by this model. In fact the owner of the ressources may cheat to set the prices as high as they can be. The way to do it is to introduce a 'fake player' who will bid $s_{i0} = (Q, p_{i0})$ where

$$i_0 \in arg \max_{i \in P} \{p_i \cdot \sum_{j \in P, p_j \geq p_i} q_j\}$$

Then the players will pay a least the price of the last served player and not at least the price of the first unserved player as it should have been.

4 An innovative method: WSP

4.1 Necessity of a new model

Having shown the weaknesses of the previous PSP-models and considering that it was an iterative method, it was obvious that a new method had to be found. The new model had to be far more efficient and stronger than the previous model since it had to tackle the problems that had been formulated. Moreover it had to avoid the issue according to which the players know each others bids. That is why we decided to propose a new model, which we will succintly present in this part.

4.2 Principle

In this method we found another approach than PSP-models. In fact instead of asking players bids each turn, the owner of the ressources collects their demand function $f : \mathbb{R}^+ \to [0,Q]$; $p_i \mapsto q_i$[2]. Having these functions for each player, the owner uses his own computation rules to set the allocation $A = (A_1, \ldots, A_n)$. The difference with this method is that the game ends after one single turn. After having set the quantities, the prices are calculated according to the "Exclusion compensative" model of Vickrey, just as in the method of N. Semret . Given these datas the name of this method becomes clearer since it reaches a **W**alrasian price equilibrium and the price are computed with the **S**econd **P**rice method.

4.3 Properties

We have already proved that our model had many interesting properties which the previous model did not have. We will succintly present you the main properties obtained thanks to his method.

4.3.1 Allocation

It has been prooved that, the way the quantities are shared out, the whole quantity of ressource is shared among players. To be more precise considering that $a_i(f)$ is the quantity the player i finally gets from the allocation, we have the relation:

[2]Gives the quantity a player would be willing to get for each price

$$\forall f \in S^n,\ \sum_{j \in \Im} a_j(f) = Q \tag{11}$$

Accordingly to this equation we may say that we avoid the problem formulated by the first model of N. Semret (cf (3.1))

4.3.2 Discount Effect

As in the PSP-model this rule brings out a discount effect, which means that the greater the quantity you get the lower is the price you pay for each unit of ressource. Let $\pi_i(f)$ the unitary price be defined by:

$$\forall i \in \Im, \forall f \in S^n,\ \pi_i(f) = \frac{c_i(f)}{a_i(f)} \tag{12}$$

We have succeded to show that we had the property below:

$$\forall i, j \in \Im,\ \forall f \in S^{n-1}\ f_i \leq h_i \Rightarrow \pi_i(f_i, f_{-i}) \geq \pi_i(h_i, f_{-i}) \tag{13}$$

This is a very important property for any single application of this model to real auctions, since every sale follows this principle. Thus the model would not be realistic if it did not have a discount effect.

4.3.3 Incentive Compatibility

Last but not least, this property[3] obtained with the WSP-model is probably the one which most distinguishes it from previous models. In fact, not only does the game reaches its end in one turn, but it also reaches a Nash Equilibrium and it respects the Incentive Compatibility. We thus have the result given by the equation (14).

$$\forall i \in \Im, \forall f \in S^n,\ u_i(w_i, f_{-i}) \geq u_i(f_i, f_{-i}) \tag{14}$$

Where w_i is the **real** demand function of player i.

5 Conclusion

We deliberately chose not to be precise in the description of the WSP-model. We had many reasons to do so, but the main of them was that the theory is being patented so we cannot reveal it before we have the official patent. However by the time of the colloquium we will be able to reveal the whole theory of this model.

[3] Every player has a real incentive to play his true demand function to have a greater gain.