Extension of Bruss' problem to chosing the best or the second best option

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Abstract. In this paper the extension of the secretary problem with random number of objects is considered. In this continuous-time generalization the objects appear according to the Poisson process with unknown intensity assumed to be exponentially distributed. The model of such stream of option has been presented by Bruss (1987). Here it is assumed that the aim of the decision maker is to choose the best or the second best candidate. The solution is compared with the asymptotic behaviour of the related problem when the deterministic number of candidates is tending to infinity.

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1. Introduction

The standard secretary problem and its versions was solved by Gilbert at al. (1966). Generalizations were presented by Ferguson (1989) and Samuels (1991). Reformulation of the sectretary problem as the optimal stopping problem of a Markov sequence is given in Dynkin at al. (1969) and Shiryaev (1978). Version of the problem for a random number of object in a finite time interval was investigated by Cowan at al. (1978). Another approach to the problem with the random number of object was presented by Presman at al. (1972). Bruss (1987) extended the model by using a compound Poisson process. Ano (2001) analyzed multiple selection for Presman and Sonin version of problem.

Bruss (1987) studied a continuous-time generalization of the secretary problem: A man has been allowed a fixed time T in which to find an apartment. Opportunities to inspect apartments occur at the epochs of a homogeneous Poisson process of unknown intensity λ. He inspects each apartment immediately when the opportunity arises, and he must decide directly whether to accept or not. At any epoch he is able to rank a given apartment amongst all those inspected to date, where all permutations of ranks are equally likely and independent of the Poisson process. The objective is to maximize the probability of selecting the best apartment from those (if any) available in the interval $[0, T]$.

In this paper we investigate a version of Bruss' problem. The difference is in the goal function. In Section 2. we formulate and in Section 3. we analyze a case when the objective is to stop on the best or the second best object (apartment). We give the optimal strategy and the value (probability of success). Relations of the asymptotic solutions (when the number of objects tends to infinity, see Gilbert at al. (1966)) is given.

2. The problem of optimal stopping for option arriving according compound Poisson process

Let $S_1, S_2, \ldots$ denote the arrival times of the Poisson process $\{N_t\}_{t\geq 0}$. For unknown intensity λ an exponential prior density $g(\lambda) = ae^{-a\lambda}\mathbb{I}_{\{\lambda>0\}}(\lambda)$ is assumed, where a is known, positive parameter. By Bayes' theorem, the conditional posterior density is of the form

$$\begin{aligned} f(\lambda|S_j = s) &= f(\lambda|S_j = s, S_{j-1} = s_{j-1}, \ldots, S_1 = s_1) \\ &= \frac{\lambda^j}{j!}(s+a)^{j+1}e^{-(s+1)\lambda}\mathbb{I}_{\{\lambda>0\}}(\lambda), s \in [0,T] \end{aligned}$$

(see Karlin (1966) and Bruss (1987)).

$$\begin{aligned} &\mathbf{P}(N(T) = n|S_1 = t_1, \ldots, S_{j-1} = t_{j-1}, S_j = s) \\ &\quad = \mathbf{P}(N(T) = n|S_j = s) = \binom{n}{j}(\frac{s+a}{T+a})^{j+1}(1-\frac{s+a}{T+a})^{n-j}. \end{aligned} \tag{1}$$

Let (j, s) denote the state of the process, when the option number j arrives at time s. Define the relative rank of the j-th option by Y_j and its absolute rank by X_j (for the details see Suchwałko at al. (2002)). Based on observation of the relative ranks and the moments of arrivals of the candidates the aim is to stop on the best or on the second best.

Let $\mathcal{F}_t = \sigma\{N_t, Y_1, Y_2, \ldots, Y_{N_t}\}$ and let $\mathfrak{M}$ be the set of all stopping times with respect to σ-fields $\{\mathcal{F}_t\}_{t\geq 0}$.

$$\mathbf{P}(X_{\tau^*} \in \{1,2\}) = \sup_{\tau\leq T}\mathbf{P}(X_\tau \in \{1,2\}) \tag{2}$$

One can consider the arrival times only and $\mathcal{F}_n = \sigma\{S_1, \ldots, S_n, Y_1, \ldots, Y_n\}$, because $\mathcal{F}_t$ for $S_n \leq t < S_{n+1}$ is equivalent with $\mathcal{F}_n$. We can consider equivalently

$$\mathbf{P}(X_{\sigma^*} \in \{1,2\}) = \sup_{\sigma}\mathbf{P}(X_\sigma \in \{1,2\}) \tag{3}$$

3. Solution of the problem of stopping on the best or on the second best

For further consideration we have $\xi_j = (j, S_j, Y_j)$. Let us define

$$W_j^r(s) = \sup_{\tau\geq j}\mathbf{P}(X_\tau \in \{1,2\}|S_j = s, Y_j = r) \tag{4}$$

and

$$U_j^r(s) = \sum_{n=j}^{\infty}\mathbf{P}(X_j \in \{1,2\}, N(T) = n|S_j = s, Y_j = r).$$

We have (see Gilbert at al. (1966))

$$\mathbf{P}(X_j \in \{1,2\}, N(T) = n | Y_j = 1) = \begin{cases} \frac{j(2n-j-1)}{n(n-1)} & \text{for } r = 1, \\ \frac{j(j-1)}{n(n-1)} & \text{for } r = 2. \end{cases} \tag{5}$$

We calculate $U_j^r(s)$ using (5) and (1):

$$\begin{aligned} U_j^r(s) &= = \begin{cases} \sum_{n=j}^{\infty} \frac{j(2n-j-1)}{n(n-1)} \binom{n}{j} (\frac{s+a}{T+a})^{j+1} (1 - \frac{s+a}{T+a})^{n-j} & \text{for } r = 1, \\ \sum_{n=j}^{\infty} \frac{j(j-1)}{n(n-1)} \binom{n}{j} (\frac{s+a}{T+a})^{j+1} (1 - \frac{s+a}{T+a})^{n-j} & \text{for } r = 2. \end{cases} \\ &= \begin{cases} \frac{s+a}{T+a}(2 - \frac{s+a}{T+a}) & \text{for } r = 1, \\ (\frac{s+a}{T+a})^2 & \text{for } r = 2. \end{cases} \end{aligned}$$

Define the probability of realizing the goal doing one step more starting from (j, s, r)

$$V_j^r(s) = \int_0^{T-s} \sum_{k=1}^{\infty} p_{(j,s,r_j)}^{(k,u,r_{j+k})} W_{j+k}^{r_{j+k}}(s+u) du \tag{6}$$

where

$$\begin{aligned} p_{(j,s,r_j)}^{(k,u,r_{j+k})} = & \int_0^{\infty} \mathbf{P}(S_{j+k} = s + u | S_j = s, \lambda) \\ & \times \mathbf{P}(Y_{j+k} = r_{j+k} | Y_j = r_j, S_j = s, S_{j+k} = s + u, \lambda) \cdot f(\lambda | S_j = s) d\lambda \end{aligned}$$

and

$$\begin{aligned} q_{r_j, r_{j+1}}(s, u, \lambda) &= \mathbf{P}(Y_{j+k} = r_{j+k} | Y_j = r_j, S_j = s, S_{j+k} = s + u, \lambda) \\ &= \begin{cases} \frac{j}{(j+k)(j+k-1)} & \text{for } r_j, r_{j+k} = 1, \\ \frac{j(j-1)}{(j+k)(j+k-1)(j+k-2)} & \text{for } r_j, r_{j+k} \in \{1,2\}. \end{cases} \end{aligned} \tag{7}$$

By the theory of optimal stopping we have

$$W_j^r(s) = \max\{U_j^r(s), V_j^r(s)\} \text{ for } j = 1, 2, \ldots, \ s \in [0, T], \ r \in \{1, 2\}. \tag{8}$$

We have (see Bruss (1987)) for $r_j, r_{j+k} = 1$

$$\begin{aligned} p_{(j,s,r_j)}^{(k,u,r_{j+k})} &= \int_0^{\infty} \frac{\lambda e^{-\lambda u} (\lambda u)^{k-1}}{(k-1)!} q_{r_j, r_{j+1}}(s, u, \lambda) \frac{e^{-\lambda(s+a)} \lambda^j (s+a)^{j+1}}{j!} d\lambda \\ &= \frac{s+a}{(s+a+u)^2} \binom{j+k-2}{k-1} (\frac{s+a}{s+a+u})^j (\frac{u}{s+a+u})^{k-1} \end{aligned} \tag{9}$$

and for $r_j, r_{j+k} \in \{1, 2\}$

$$\begin{aligned} p_{(j,s,r_j)}^{(k,u,r_{j+k})} &= \int_0^{\infty} \frac{\lambda e^{-\lambda u} (\lambda u)^{k-1}}{(k-1)!} q_{r_j, r_{j+1}}(s, u, \lambda) \frac{e^{-\lambda(s+a)} \lambda^j (s+a)^{j+1}}{j!} d\lambda \\ &= \frac{s+a}{(s+a+u)^2} \binom{j+k-3}{k-1} (\frac{s+a}{s+a+u})^j (\frac{u}{s+a+u})^{k-1}. \end{aligned} \tag{10}$$

Let B be the one-step look-ahead stopping region. It means that B is the set of states (j, s) for which selecting the current relatively best or second best option is at least as good as waiting for the next relatively best or second best option to appear and then selecting it. Define additionally the average payoff for doing one step more by

$$R_j^r(s) = \int_0^{T-s} \sum_{t=1}^{2} \sum_{k=1}^{\infty} p_{(j,s,r)}^{(k,u,t)} U_{j+k}^t(s+u)du \tag{11}$$

Therefore the set B is given by formula

$$B = \{(j, s, r) : U_j^r(s) - R_j^r(s) \geq 0, r = 1, 2\} \tag{12}$$

This set can be divided into two parts

$$B = B_1(\alpha, \beta) \cup B_{1,2}(\beta, 1) = \{\} \cup \{\} \tag{13}$$

In order to find the set $B_{1,2}(\beta, 1)$ we are solving the inequality from (12). Let us define

$$h_j^r(s) = U_j^r(s) - R_j^r(s) = \begin{cases} (\frac{s+a}{T+a})^2 & \text{for } r = 1, \\ 3(\frac{s+a}{T+a})^2 - 2\frac{s+a}{T+a} & \text{for } r = 2. \end{cases}$$

Then

$$B_{1,2}(\beta, 1) = \{(j, s, t) : t \in \{1, 2\}, s \geq s^*\},$$

where

$$s^* = \min\{s : \frac{s+a}{T+a} \geq \frac{2}{3}\}. \tag{14}$$

We have $\beta = \frac{2}{3}$. To find the set $B_1(\alpha, \beta)$ we calculate $h_j^1(s)$ for $s \leq s^*$. We have

$$\begin{aligned} h_j^1(s) = {} & \frac{s+a}{T+a}(2 - \frac{s+a}{T+a}) - 2\frac{s+a}{T+a} \ln \frac{s^*+a}{s+a} \\ & + \frac{s+a}{(T+a)^2}(s^* + a - (s+a)) - 2\frac{s+a}{s^*+a}\frac{s^*+a}{T+a}(1 - \frac{s^*+a}{T+a}). \end{aligned}$$

For $x = \frac{s+a}{T+a}$ and $y = \frac{s^*+a}{T+a}$ we get that $h_j^1(s)$ takes the form

$$h_j^1(s) = x(2-x) - 2x \ln \frac{y}{x} + x(y-x) - 2x(1-y). \tag{15}$$

From the definition of s^* (see (14)) we have $y = \beta = \frac{2}{3}$. Solving the equation (15) we get $x = \alpha \cong 0.347$.

4. Conclusion

We have given the optimal strategies for a version of the Bruss' problem. The optimal strategy has the threshold form. There are similarities between optimal strategy for this problem and the secretary problem with number of objects tending to infinity. The constants α and β are equal.

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