On the asymptotics of big components in a generalized allocation scheme

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Let us consider non-negative integer random variables $\eta_1, \ldots, \eta_N$ with a joint distribution defined by the sequence $b = (b_0, b_1, b_2, \ldots)$ and the parameter n as follows. For all integer non-negative numbers $k_1, \ldots, k_N$ which sum is equal to n we have

$$\mathbf{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = \frac{b_{k_1} \cdots b_{k_N}}{\sum\limits_{j_1+\cdots+j_N=n} b_{j_1} \cdots b_{j_N}}. \tag{1}$$

In [1] it was showed that probabilities (1) can be represented as

$$\mathbf{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = \mathbf{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N |\, \xi_1 + \cdots + \xi_N = n\}, \tag{2}$$

where $\xi_1, \xi_2, \ldots$ are independent identically distributed random variables such that

$$\mathbf{P}\{\xi_1 = k\} = x^k b_k / B(x), \quad B(x) = \sum_k x^k b_k,$$

and a parameter x depends on N, n in such a way that simplify an obtaining corresponding local limit theorems for sum of random $\xi_1, \ldots, \xi_N$ (usually we determine x from the equation $\mathbf{E}\,\xi_1 = n/N$) [2].

If $\eta_1, \ldots, \eta_N, \xi_1, \ldots, \xi_N$ satisfies (2) then we say that these random variables forms the generalized allocation scheme [1]. In books [1, 3, 4] one can find several examples of combinatorial tasks which can be reduced to generalized allocation scheme. It allow us to resolve these tasks by using asymptotical methods of Probability theory when $N \to \infty$. Last investigations [4, 5] shows an actuality of such approach.

Let $\eta_{(1)} \leqslant \ldots \leqslant \eta_{(N)}$ be a variational series of random variables $\eta_1, \ldots, \eta_N$. If $p = o(N)$ as $N \to \infty$ then the component $\eta_{(N-p)}$ is called big. Recently an interest to research of limit distributions of big components of generalized allocation scheme is being enhanced [6, 7].

Further we suppose that the support of b has maximum span 1, $b_0 > 0$ and radius of convergence of $\sum_k x^k b_k$ equals to 1. By $\mathfrak{B}_N(\alpha)$ we denote a binomial random variable with parameters N and α. For non-negative integer r denote

$$P_r = \sum_{k>r} x^k b_k / B(x).$$

Let m be minimal natural number such that $\gcd(\mathrm{supp}(b_0, \dots, b_m)) = 1$. Below symbols $C_1, C_2, \dots$ denotes some positive constants.

Theorem 1. Let $N, n \to \infty$ in such a way that $n/N \leqslant C_1 < \sup_x \mathbf{E}\,\xi_1$. Let also the parameter x satisfies the equation $n/N = \mathbf{E}\,\xi_1$, and following conditions are valid:
(a) $b_k \leqslant C_2 b_l$ for all k and all $l < k$ such that $b_l > 0$; besides as $k \geqslant k_0$ we have $b_k > 0$;
(b) $Nx^m \to \infty$; $0 < x \leqslant C_3 < 1$;
(c) $p^3 = o(N\,\mathbf{D}\,\xi_1)$;
(d) $C_4(p+1) \leqslant NP_r \leqslant C_5(p+1)$; $r \geqslant m$, $b_r > 0$;
(e) $C_6 \leqslant b_k/b_{k+1} \leqslant C_7$ as $k \geqslant k_0$.
Then
(A) for $n/N \geqslant C_8$ and any fixed integer h

$$\mathbf{P}\{\eta_{(N-p)} \leqslant r + h\} = (1 + o(1))\,\mathbf{P}\{\mathfrak{B}_N(P_{r+h}) \leqslant p\},$$

(B) as $n = o(N)$ we have

$$\mathbf{P}\{\eta_{(N-p)} = r\} = (1 + o(1))\,\mathbf{P}\{\mathfrak{B}_N(P_r) \leqslant p\};$$
$$\mathbf{P}\{\eta_{(N-p)} = r^+\} = (1 + o(1))(1 - \mathbf{P}\{\mathfrak{B}_N(P_r) \leqslant p\}),$$

where $r^+ = \min\{k|\ k > r, b_k > 0\}$.

¿From this theorem follows some known results about asymptotics of distributions of components $\eta_{(N-p)}$ of Galton—Watson forest [5] as p is fixed. Besides this theorem give us analogous results for $p \to \infty$ non-rapidly (see the condition (c)), and an asymptotics of big components of random recursive forest for $n = O(N)$.

Theorem 1 does not give us full description of limit distribution of big components because from the condition $n/N \leqslant C_1$ it follows that the parameter x does not approach to a critical point — radius of convergence of $B(x)$.

Currently it becomes obviously to create a common theory that would describe a behaviour of components of a generalized allocation scheme (2) in all zones of range of parameters N, n with sufficiently weak conditions for the sequence b.

The research was supported by the Russian Foundation for Basic Research, grant 00–01–00233.

References

1. Kolchin V. F. Random Mappings. Springer, New York, 1986.

2. Kazimirov N. I. One case of local limit theorem. // Proc. Institute App. Math. Research of KRC RAS, V.3. Petrozavodsk, 2002.

3. Kolchin V. F., Sevastyanov B. A., Chistyakov V. P. Random Allocations. Wiley, New York, 1978.

4. Kolchin V. F. Random Graphs. Fizmatlit, Moscow, 2000.

5. Pavlov Yu. L. Random Forests. VSP, Utrecht, 2000.

6. Pavlov Yu. L., Loseva E. A. Limit distributions of maximum tree size in a random recursive forest. // Discr. Math, 2002. (in print)

7. Kazimirov N. I. On some conditions of absence of a giant component in a generalized allocation scheme. // Discr. Math, 2002. (in print)