Competitive Flow Control in General Topology Networks *

E. Altman, R. El Azouzi
INRIA, BP 93, 2004 route des Lucioles,
06902 Sophia-Antipolis Cedex, France.
{altman,relazouz}@sophia.inria.fr

T. Başar and R. Srikant
Coordinated Science Laboratory
University of Illinois, 1308 West Main Street, Urbana, IL 61801-2307, USA.
tbasar@control.csl.uiuc.edu , rsrikant@uiuc.edu

Abstract

We consider in this paper the problem of flow control where the routing strategies are fixed and the users control only their flows in a noncooperative setting. Modeling this as a networking game, we obtain results on the existence and uniqueness of Nash equilibrium in a general topology setting. We also obtain, using the theory of S-modular games [15], convergence results associated with a best response greedy algorithm in a network of two users, again with general topology.

1 Introduction

Flow control is part of resource and traffic management in today's high-speed networks, such as the Internet and the ATM. Flow control is used by best-effort type traffic in order to adjust the input transmission rates (the instantaneous throughput of a connection) to the available bandwidth in the network. An appropriate framework for modeling flow control in the case of selfish users and in the absence of a network centralized control is that of non-cooperative game theory.

Flow control in a non-cooperative game setting has been studied in the past in settings in which each user seeks to obtain optimal tradeoff between expected delay and throughput. In particular, the authors of [8] study the problem in which each user seeks to maximize its expeced throughput subject to a bound on the expected delay. Another way to express the tradeoff between delay and throughput is the power criterion, which

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is the ratio between throughput (or some power of it) and expected delay. Examples of work on flow control games using that (or related) criterion are [2, 4, 5, 11, 12, 13, 16].

A different dynamic game with another type of a criterion has been studied in [1]. The objective of each user was to track a desirable queue length on the one hand (so as to avoid losses of packets when queue builds up, or to avoid losing in throughput when the queue empties) and to track well the system available bandwidth, on the other.

In this paper we use an alternative general cost crietrion per user which is the sum of a costs (or delays) incurred at the network and some utility for the attained throughput. This is in line with more traditional approach for flow control (as part of combined flow and routing control); see [6, 10]. Although flow control has been addressed in the past in that framework [6], conditions for uniqueness of equilibria have not been obtained previously. This is our first objective in this paper. In particular, we are interested in obtaining uniqueness results for cases which are more general than a single link (for which no such results are available).

Subsequently, we study the question of convergence to the equilibrium and show, using the theory of S-modular games [15], that a dynamic asynchronous greedy updating scheme converges to the unique equilibrium. These specific results extend some previous results on convergence to equilibrium in the context of noncooperative control of flow, such as those in [1, 4, 16].

The paper is structured as follows. In the next section, we introduce the model and the underlying assumptions. In Section 3, we study the flow control problem in a single link and also obtain some results for networks with a general topology. We conclude the paper with Section 4, which discusses the question of convergence to the equilibrium.

2 The model

We consider a network with general topology $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ where $\mathcal{N}$ is a finite set of nodes and $\mathcal{L}$ is a set of links. We consider an extension of directional links (see [3]) where a link may carry traffic in both directions, but the direction for each user is fixed. We consider a set $\mathcal{I}$ of I selfish users (players) who share the network $\mathcal{G}$. User $i \in \mathcal{I}$ has a source $s(i)$ and a destination $d(i)$. Each user has to determine the amount of flow $r^i \in R^i := [m^i, M^i]$ to ship between $s(i)$ and $d(i)$ with routing decisions already taken and fixed.

Definition 2.1. *A problem in which the routing decisions are fixed and do not depend on the flow is said to be a competitive flow control problem. By fixed routing decisions we mean that for each player i, the proportion $x^i_l := f^i_l/r^i$ of flow sent over any link $l \in \mathcal{L}$ is fixed, where f^i_l is the amount of flow that user i sends over link l, and r^i is the total flow of user i. Without loss of generality we assume that $x^i > 0$ $\forall i$ where $x^i = \sum_{l\in\mathcal{L}} x^i_l$.*

Let $J^i(\mathbf{r})$ be the cost for user i when the vector of flows by all users is given by $\mathbf{r}$. We seek a Nash Equilibrium (NE) $\tilde{\mathbf{r}}$, that is an $\tilde{\mathbf{r}}$ satisfying

$$J^i(\tilde{\mathbf{r}}) = \min_{r^i \in \mathbf{R^i}} J^i(\tilde{r}^1, ..., \tilde{r}^{i-1}, r^i, \tilde{r}^{i+1}, ...r^I). \tag{1}$$

We make the following assumptions on the cost function J^i for user i, which will be invoked throughout the paper:

G1 J^i is given as the sum of link costs $J^i_l(\mathbf{f_l})$ minus the utility function $U^i(r^i)$:

$$J^i(\mathbf{r}) = \sum_{l \in \mathcal{L}} \hat{J}^i_l(\mathbf{f_l}) - U^i(r^i).$$

where $\mathbf{f}_l = (f^1_l, .., f^I_l)$ with $f^i_l = x^i_l r^i$.

G2 $\hat{J}^i_l : [0, \infty)^I \to [0, \infty]$ is continuous.

G3 $\hat{J}^i_l$ is convex in f^i_l and U^i is concave in its argument.

G4 Whenever finite, $\hat{J}^i_l$ are continuously differentiable in f^i_l. We let $K^i_l := \partial \hat{J}^i_l / \partial f^i_l$, $l \in \mathcal{L}$ and $K^i_0 = -\partial U^i(r^i)/\partial r^i$.

G5 For every system flow configuration $\mathbf{f}$, if not all costs are finite, then at least one user with infinite cost ($J^i(\mathbf{f}) = \infty$) can change its own flow configuration to make its cost finite.

Existence of a Nash Equilibrium

We first show that a Nash equilibrium exists. The link cost function for each user i is convex and continuous in r^i (see **G3** and note that $f^i_l = r^i x^i_l$), and the game strategy space is convex, closed, and bounded set. Hence the competitive flow control (1) with cost functions of type **G** is equivalent to a convex game in the sense of [14], thus the existence of a **NE** is guaranteed [14, Thm. 1].

We end this section by introducing some further relatively weak assumptions on the cost functions, that will allow us to establish in the following sections uniqueness of the Nash equilibrium. Our first set of assumptions is the following:

A1 $\hat{J}^i_l$ is a function of two arguments: user i's flow on link l and the total flow on that link. In other words, $\hat{J}^i_l(\mathbf{f_l}) = \hat{J}^i_l(f^i_l, f_l)\,,\ \ i \in \mathcal{I}, l \in \mathcal{L}$.

A2 $\hat{J}^i_l$ is increasing in each of its arguments, and U^i is an increasing function of its argument,

A3 Viewing $K^i_\ell = K^i_\ell(f^i_\ell, f_\ell)$ now as a function of two arguments, whenever J^i_l is finite, $K^i_l(f^i_l, f_l)$, $l \in \mathcal{L}$, is increasing in each of its two arguments, and strictly increasing in the first one. Moreover, K^i_0 is strictly increasing in its argument, all for $i \in$.

As in [9], we refer to functions that comply with these three assumptions as type-A functions.

Our second set of assumptions is the following: For each $i \in \mathcal{I}, l \in \mathcal{L}$,

B1 $J^i(\mathbf{f}) = \sum_{l\in\mathcal{L}} f_l^i T_l(\mathbf{f_l})$

B2 $T_l : [0,\infty) \to (0,\infty]$

B3 $T_l(\mathbf{f_l})$ is positive, strictly increasing and convex.

B4 $T_l(\mathbf{f_l})$ are continuously differentiable.

Remark 2.1. *For fixed proportions of flows with N noncoperative users where $N > 2$, we can write the cost functions of type* **A** *or* **B** *for user i as*

$$J^i(\mathbf{r}) = J^i(r^i, r) \tag{2}$$

if all users have the same proportion of flow in each link, i.e., $x_l^j := x_l \ \forall l \in \mathcal{L}, j \in \mathcal{I}$. Indeed, we have

$$J^i(\mathbf{r}) = \sum_l \hat{J}_l^i(f_l^i, f_l) - U^i(r^i) = \sum_l \hat{J}_l^i(r^i x_l, r x_l) - U^i(r^i) =: J(r^i, r). \tag{3}$$

3 Uniqueness of Nash equilibrium

We first establish uniqueness of Nash equilibrium in a single link topology. Toward this end, we take the cost function of user i to be in the form:

$$J^i(r^i, r) = \hat{J}_{sl}^i(r^i, r) - U^i(r^i) \tag{4}$$

where '*sl*' stands for 'single link,' and r is the total flow of all users. We further assume that the following condition holds:

Condition C: $J^i(r^i, r)$ is increasing in its first argument, and its partial derivative with respect to r^i is increasing in both arguments and strictly increasing in the first one.

Then we have:

Lemma 3.1. *Consider a system of N noncooperative users sharing a single link where the cost functions are of type* **A**, *and condition* **C** *holds. Then, the Nash equilibrium is unique.*

Proof: The proof of lemma follows readily from the more general uniqueness result obtained in [11] for parallel links. ∎

Remark 3.1. *For $m^i = 0 \ \forall i$, we can use another proof to show general uniqueness in parallel links by introducing the following equivalent network with fixed demand:*

Equivalent network*: Consider a network $\hat{\mathcal{G}}(\mathcal{V}, \hat{\mathcal{L}})$ where $\mathcal{V}$ is the same set of nodes as in the original network, and where $\hat{\mathcal{L}} := \mathcal{L} \cup \{l_0\}$ is the set of directed links, where l_0 is a new link between $s(i)$ and $d(i)$ (which are the same as in the original network). In this*

network, each user i has a fixed demand of $\overline{r}^i =: M^i$ to ship from $s(i)$ to $d(i)$. For $l = l_0$, $i \in \mathcal{I}$, we define $f_l^i := \overline{r}^i - r^i$.
This is thus a routing game for a fixed demand, with no flow control. The cost J^i for user i is given as the sum of link costs $J_l^i(\mathbf{f_l})$ over all $l \in \hat{\mathcal{L}}$, where we define for $l = l_0$, and each $i \in \mathcal{I}$,

$$J_l^i(\mathbf{f_l}) = -U^i(r^i) = -U^i(\overline{r}^i - f_l^i),$$

The uniqueness of a Nash equilibrium in the case of parallel links topology now readily follows from [9, Thm 1]:

We can get dispense with condition **C** if there are only two users, and the cost functions are of type **B**.

Lemma 3.2. *Consider a system of two noncooperative users sharing a single link sl where the cost functions are of type* **B**. *Suppose that $x_{sl}^i x_{sl}^j > 0$. Then, the Nash equilibrium is unique.*

Proof: We rewrite the cost function of user i as

$$J^i(\mathbf{r}) = J^i(r^i, r^j) = \hat{J}_{sl}(r^i, r^i) - U^i(r^i)$$

A unique Nash equilibrium $\mathbf{r}$ is guaranteed, by Rosen [14], if the following matrix

$$G_l := \left\{ \frac{\partial^2 J_{sl}^i(r^i, r^j)}{\partial r^i \partial r^j} \right\}_{i,j \in \{1,2\}} \tag{5}$$

is positive definite (note that the mixed derivatives of the utility part are zero). A set of sufficient conditions for G_{sl} to be positive definite is

1. $a_{ii} := \frac{\partial^2 J_{sl}^i(r^i, r^j)}{\partial {r^i}^2} > 0$

2. $a_{ii}a_{jj} := \frac{\partial^2 J_{sl}^i(r^i, r^j)}{\partial {r^i}^2} \frac{\partial^2 J_{sl}^j(r^j, r^i)}{\partial {r^j}^2} > a_{ij}a_{ji} := \frac{\partial^2 J_{sl}^i(r^i, r^j)}{\partial r^i \partial r^j} \frac{\partial^2 J_{sl}^j(r^j, r^i)}{\partial r^j \partial r^i}$ $i \neq j$.

Note that

$$a_{ii} = 2(x_{sl}^i)^2 T_{sl}'(x_{sl}^i r^i + x_{sl}^j r^j) + (x_{sl}^i)^3 r^i T''(x_{sl}^i r^i + x_{sl}^j r^j) > 0$$

The last inequality follows from $x_{sl}^i x_{sl}^j > 0$ and **B2**. For parts (2) we have

$$\begin{aligned} a_{ii}a_{jj} - a_{ij}a_{ji} &= \Big(2(x_{sl}^i)^2 T_{sl}'(f_{sl}) + (x_{sl}^i)^3 r^i T''(f_{sl})\Big)\Big((x_{sl}^j)^2 T_{sl}'(f_{sl}) + (x_{sl}^i)^3 r^i T''(f_{sl})\Big) \\ &\quad - \Big(x_{sl}^i x_{sl}^j T'(f_{sl}) + (x_{sl}^i)^2 x_{sl}^j r^i T''(f_{sl})\Big)\Big((x_{sl}^j x_{sl}^i T'(f_{sl}) + (x_{sl}^j)^2 x_{sl}^j r^j T''(f_{sl})\Big) \\ &= 3(x_{sl}^i)^3 (x_{sl}^i)^2 (T'(f_{sl}))^2 + ((x_{sl}^i)^3 (x_{sl}^j)^2 r^i + (x_{sl}^j)^3 (x_{sl}^i)^2 r^j) T'(f_{sl}) T''(f_{sl}) > 0. \end{aligned}$$

This establishes the proof.

Before presenting uniqueness results in general topology networks, we consider a new equivalent transformed network.

Definition 3.1. (A single equivalent link) *Consider a competitive flow control problem with fixed routing proportions x_l^i. We define the single equivalent model to be a new isolated link shared by N users such that the cost for user i is given by the expression in (2) for $J^i(r^i, r)$. In this equivalent link, both users have the same source-destination pair.*

Proposition 3.1. *Consider a network with a general topology where the cost function of each user is of type* **A** *(with possibly different sources and destinations). Assume that condition* **C** *holds. Then the Nash equilibrium for the competitive flow control model is unique.*

Proof: We may consider the entire network as the equivalent single link given in Definition 3.1. We note that for this new link, assumption A still holds for the equivalent link. In particular, we see from (2) that $\hat{J}^i$ in the equivalent link is indeed a function of only r^i and r, as required by assumption A1. Assumptions A2 and A3 hold for the equivalent link due to condition C of the Proposition and since it holds by assumption for each link of the original model. Also, it is easily seen from (2) that $\hat{J}^i$ is increasing in its second argument as required in A2. We have thus transformed the entire network into one with a single link with type-A cost for all users, by keeping the costs unchanged. The uniqueness of the Nash equilibrium now follows from Lemma 3.1.

■

Lemma 3.3. *A sufficient condition for condition* **C** *in Proposition 3.1 to hold is E1: All users have the same proportions of flows in each link, i.e., $x_l^i =: x_l \ \forall i, l$.*

Proof:
We check that $\overline{J}^i(r^i, r) := \sum_{l\in\mathcal{L}} \hat{J}_l^i(f_l^i, f_l)$ is monotone in the first argument. Since $f_l = x_l r$, we have

$$\frac{\partial \overline{J}^i(r^i, r)}{\partial r^i} = \sum_{l\in\mathcal{L}} \frac{\partial f_l^i}{\partial r^i} \frac{\partial \hat{J}_l^i(f_l^i, f_l)}{\partial f_l^i} = \sum_{l\in\mathcal{L}} x_l K_l^i(f_l^i, f_l) = \sum_{l\in\mathcal{L}} x_l K_l^i(x_l r^i, x_l r).$$

By an earlier assumption, this is nonnegative. This implies that $\overline{J}^i(r^i, r)$ is monotone in its first argument as well. Furthemore, since there exists a link $l_0 \in \mathcal{L}$ such that $x_{l_0} > 0$, and the derivative function K_l^i is increasing in each of its two arguments, and strictly increasing in the first one, we conclude that the partial derivative $\frac{\partial \overline{J}^i(r^i,r)}{\partial r^i}$ is strictly increasing in its first component and increasing in its second component. ■

Next we develop a uniqueness results for general topology networks without condition **C**. Only the special case of two users is considered.

Proposition 3.2. *Consider a system of two users where the cost function of each user is of type* **B**. *Suppose that*

E2: There exists a link l such that $x_l^i x_l^j > 0$ for all $i, j \in I$.
Then, the Nash equilibrium is unique.

Proof: We may consider the entire network as the equivalent single link in Definition 3.1. It is then relatively easy to show that the matrix (5) is still positive definite. ∎

Remark 3.2. *Note that E2 above is not restrictive. Indeed, if E2 does not hold, then there is no interaction between the players, and each player faces an independent convex optimization problem. Due to the strict convexity over the compact set of actions of each player, if E2 does not hold then we have the unique (trivial) Nash equilibrium for the game, which is made up of solutions to independent optimization problems.*

4 Convergence to equilibrium

Finally, we consider the question of convergence of a greedy best response algorithm to the Nash equilibrium in the case of competitive flow control (for fixed routing). We restrict the analysis to two players. Update opportunities are assumed to occur at some increasing sequence of times T_n. User 1 is assumed to update at odd instants (i.e. $T_1, T_3, ...$) whereas user 2 updates at even instants (i.e. $T_2, T_4, ...$). At each opportunity, the player uses the best response to the current policy of the other user. Note that this allows for any arbitrary asynchronous update sequences for the two users, since if user i updates several times between two consecutive updates of the other user, then only the first update produces a change in user i's policy.

Proposition 4.1. *Consider a competitive flow control problem with general topology, with two users having costs of type-A. Assume that*

$$\left.\frac{\partial^2 \hat{J}_l^i(r^i x_i^l, s)}{\partial s^2}\right|_{s=f_l} + \left.\frac{\partial^2 \hat{J}_l^i(s_1, s_2)}{\partial s_1 \partial s_2}\right|_{s_1 = x_l^i r^i, s_2 = f_l} > 0 \tag{6}$$

Then the best response greedy algorithm converges to an equilibrium.

Proof: We now use the representation of the cost of the equivalent link. We rewrite it as

$$\tilde{J}^i(r^i, r^j) = \sum_l \hat{J}_l^i(r^i x_l^i, r^i x_l^i + r^j x_l^j).$$

We shall show that

$$\frac{\partial^2 \tilde{J}^i(r^i, r^j)}{\partial r^i \partial r^j} \geq 0. \tag{7}$$

(Note that the mixed derivatives of the utility part are zero.) This will be sufficient for the desired result, because from it we can conclude that the underlying game is a submodular one, implying that the greedy algorithm converges to the Nash equilibrium [15, Thm. 2.3].

Using $f_l^i = r^i x_l^i + r^j x_l^j$, we have

$$\frac{\partial \hat{J}_l^i(r^i x_l^i, r^i x_l^i + r^j x_l^j)}{\partial r^j} = \left.\frac{\partial \hat{J}_l^i(r^i x_i^l, s)}{\partial s}\right|_{s=f_l} x_l^j.$$

Further differentiating with respect to r^i, we obtain

$$\frac{\partial^2 \hat{J}_l^i(r^i x_l^i, r^i x_l^i + r^j x_l^j)}{\partial r^j \partial r^i} = x_l^j x_l^i \left[\left.\frac{\partial^2 \hat{J}_l^i(r^i x_i^l, s)}{\partial s^2}\right|_{s=f_l} + \left.\frac{\partial^2 \hat{J}_l^i(s_1, s_2)}{\partial s_1 \partial s_2}\right|_{s_1=x_l^i r^i, s_2=f_l} \right]$$

By (6), we conclude that (7) holds, which concludes the proof. ∎

Remark 4.1. *It is easy to see that: A sufficient condition for (6) to hold is that both users have a B-type cost. Indeed, in that case*

$$\frac{\partial^2 \hat{J}_l^i(r^i x_i^l, s)}{\partial s^2} = r^i x_l^i \frac{dT_l(s)^2}{ds^2} \geq 0,$$

and

$$\frac{\partial^2 \hat{J}_l^i(s_1, s_2)}{\partial s_1 \partial s_2} = \left.\frac{dT_l(s)}{ds}\right|_{s=s_1+s_2} + s_1 \left.\frac{d^2 T_l(s)}{ds^2}\right|_{s=s_1+s_2} \geq 0.$$

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