# Game-theoretic analysis of network structure

Nikolay Ermolin, Vassiliy Gusev, Alexey Kondratev, Vladimir Mazalov, Nalalia Nikitina

Institute of Applied Mathematical Research
Karelian Research Centre of the Russian Academy of Sciences

## Introduction

The betweenness centrality is one of the basic concepts in the analysis of the social networks. Initial definition for the betweenness of a node in the graph is based on the fraction of the number of geodesics (shortest paths) between any two nodes that given node lies on, to the total number of the shortest paths connecting these nodes. This method has polynomial complexity. We propose a new concept of the betweenness centrality for weighted graphs using the methods of cooperative game theory. The characteristic function is determined by special way for different coalitions (subsets of the graph). Two approaches are used to determine the characteristic function. In the first approach the characteristic function is determined via the number of direct and indirect weighted connecting paths in the coalition. In the second approach the coalition is considered as an electric network and the characteristic function is determined as a total current in this network. We use the Kirchhoff's law. After that the betweenness centrality is determined as the Myerson value. The results of computer simulations for some examples of networks are presented.

## Kirchoff centrality

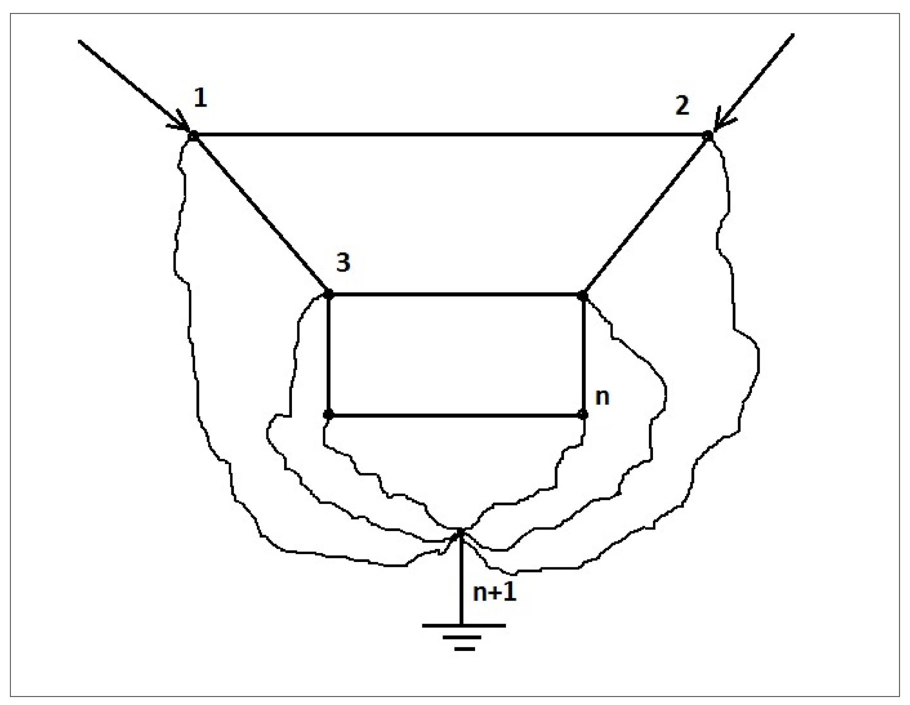
Consider a weighted graph $G = (V, E, W)$, where $w_{ij} \geq 0$ is weight of the edge connecting the nodes $i$ and $j$. $d_i = \sum_{j=1}^{n} w_{i,j}$ is the sum of weights of the edges which are adjacent to node $i$ in graph $G$. The Laplacian matrix $L(G)$ for weighted graph $G$ is defined as follows:

$$L(G) = D(G) - W(G) = \begin{pmatrix} d_1 & -w_{1,2} & \dots & -w_{1,n} \\ -w_{2,1} & d_2 & \dots & -w_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n,1} & -w_{n,2} & \dots & d_n \end{pmatrix}.$$

Suppose that a unit of current enters into the node $s \quad V$ and the node $n + 1$ is grounded. Let $\varphi_i^s$ be the electric potential at node $i$ when an electric charge is located at node $s$. The vector of all potentials $\varphi^s(G) = [\varphi_1^s, \dots, \varphi_n^s, \varphi_{n+1}^s]^T$ for the nodes of graph $G$ is determined by the following system of equations (Kirchhoff's current law):

$$L(G)\varphi^s(G) = b_s,$$

where $b_s$ is the vector of $n + 1$ components with the values: $b_s(i) = 1$ if $i = s$, and 0 otherwise.




(a) Kirchoff centrality is the mean value of the current passing through node

## Community detection as a cooperative game

### Potential games

Consider a cooperative game with coalition structure $N, v, \pi$, $\pi$ gives a coalition partition of the set $N$, i.e., $\pi = \{B_1, B_2, ..., B_l\}$, where $\bigcup_{j=1}^{l} B_j = N$ and $i, j \quad \{1, 2, ..., l\}, i = j : B_i \cap B_j =$ .

Let $x_i(N, v, \pi)$ be the payoff or value of player $i$. Assume, that a value $x(N, v, \pi)$ is satisfied component independence, i.e. $x_i(N, v, \pi) = x_i(B(i), v, \pi)$ and $x_i(N, v, \pi)$ does not depend on $v(K), K \quad N \setminus B(i), \quad i \quad N$.

Let $D_i(\pi)$ is the set of the coalition structures obtained from $\pi$ if player $i$ moves to another coalition or forms a coalition himself.

**Definition 1.** *A coalition structure $\pi$ is said to be Nash-stable if $i \quad N : x_i(N, v, \pi) \geq x_i(N, v, \rho), \quad \rho \quad D_i(\pi)$ for a given value $x(N, v, \pi)$.*

**Definition 2.** *A function $P(N, v, \pi)$ will be called a potential function in a cooperative game with coalition structure $N, v, \pi$ if $i \quad N, \quad \pi \quad \Pi(N)$ :*

$$x_i(N, v, \pi) - x_i(N, v, \rho) = P(N, v, \pi) - P(N, v, \rho), \quad \rho \quad D_i(\pi)$$

*for a given value $x(N, v, \pi)$.*

**Theorem 1 (Gusev, Mazalov 2020).** *A cooperative game $N, v, \pi$ is a potential game and the coalition partition $\pi$ giving a local maximum of the potential is the Nash-stable partition.*

### Hedonic games

Consider a game with hedonic coalitions, where a player's payoff is completely determined by the identity of other members of his coalition. Assume that the set of players $N = \{1, \dots, n\}$ is divided into $K$ coalitions: $\Pi = \{S_1, \dots, S_K\}$. Let $S_\Pi(i)$ denote the coalition $S_k \quad \Pi$ such that $i \quad S_k$. A player $i$ preferences are represented by a complete, reflexive and transitive binary relation $\ _i$ over the set $\{S \quad N : i \quad S\}$. The preferences are additively separable if there exists a value function $v_i : N \to \mathbb{R}$ such that $v_i(i) = 0$ and

$$S_1 \ _i S_2 \Leftrightarrow \sum_{j \ S_1} v_i(j) \geq \sum_{j \ S_2} v_i(j).$$

The preferences $\{v_i, i \quad N\}$ are symmetric $v_i(j) = v_j(i) = v_{ij} = v_{ji}, \quad i, j \quad N$.

**Definition 3.** The network partition $\Pi$ is Nash stable, if $S_\Pi(i) \ _i S_k \quad \{i\}$ for all $i \quad N, S_k \quad \Pi \quad \{ \}$. In the Nash-stable partition, there is no player who wants to leave her coalition.

A potential of a coalition partition $\Pi = \{S_1, \dots, S_K\}$ is

$$P(\Pi) = \sum_{k=1}^{K} P(S_k) = \sum_{k=1}^{K} \sum_{i,j \ S_k} v_{ij}.$$

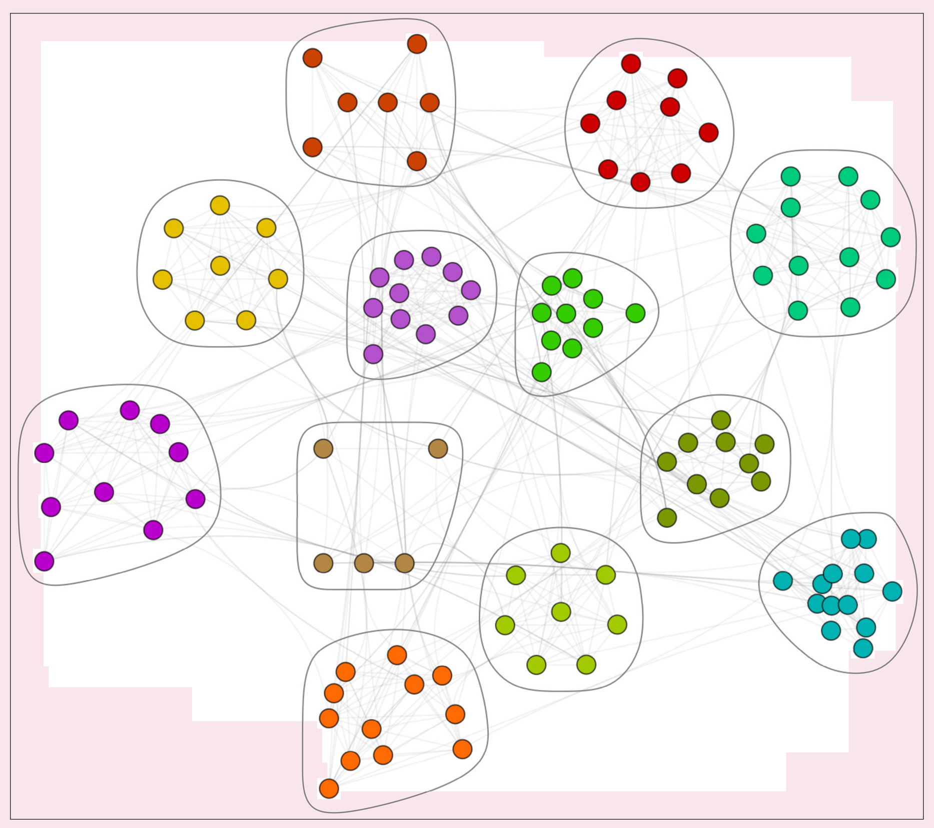
Define a value function $v$ with a parameter $\alpha \quad [0, 1]$ is as follows:

$$v_{ij} = \begin{cases} 1 - \alpha, & (i, j) \quad E, \\ -\alpha, & (i, j) \ / \ E, \\ 0, & i = j. \end{cases}$$

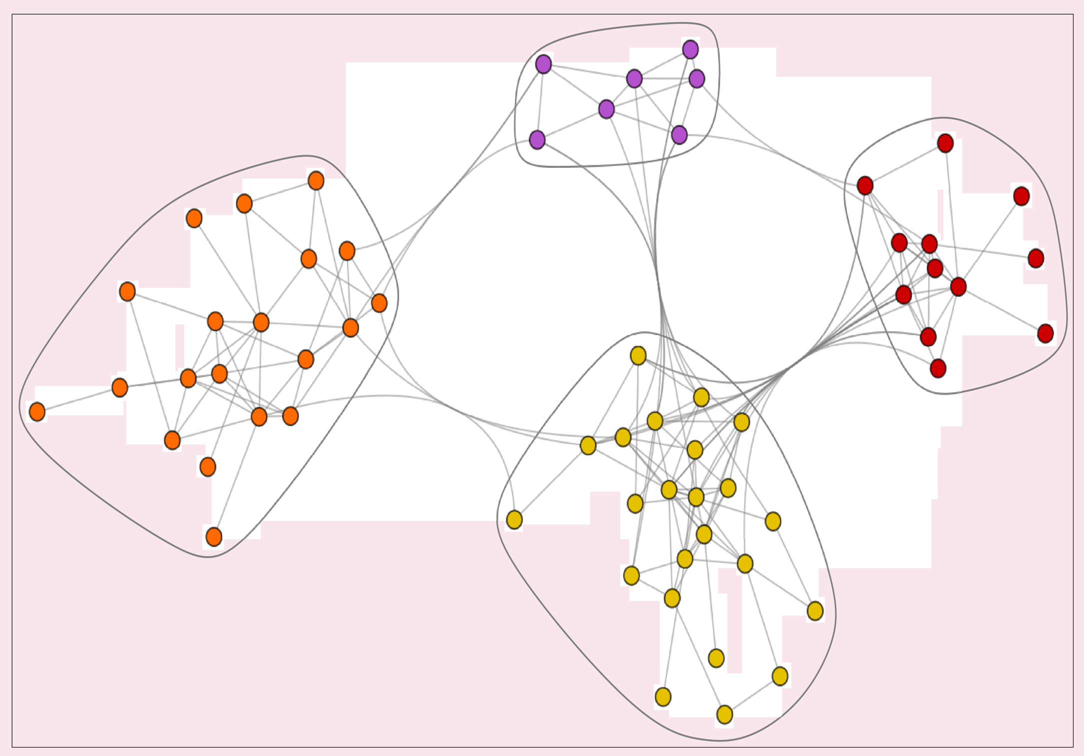
For any subgraph $(S, E|S)$, $S \quad N$, denote $n(S)$ as the number of nodes in $S$, and $m(S)$ as the number of edges in $S$. Then,

$$P(\Pi) = \sum_{k=1}^{K} \left( m(S_k) - \frac{n(S_k)(n(S_k) - 1)\alpha}{2} \right).$$

**Theorem 2 (Avrachenkov, et al.2018)** *The coalition partition $\Pi$ giving a local maximum of the potential $P(\Pi)$ is the Nash-stable partition.*



(b) Football network.



(c) Dolphins social network.

## Game-theoretic centrality measures

Graph $g = < N, E >$. Determine characteristic function: $v(K)$ is the number of simple paths of fixed length $m$ in subgraph $g_K$. Let centrality of node $i$ is the Myerson value in cooperative game.

**Theorem 3 (Mazalov, et al. 2014, Avrachenkov, et al. 2018).** The Myerson value for player $i$ can be calculated as

$$M_i^m = a(i)/(m + 1),$$

where $a(i)$ denotes the number of simple paths of length $m$ passing through vertex $i$.

Consider a cooperative game (packing game) with characteristic function $v^G(K)$ as the maximum number of minimal winning coalitions:

$$v^G(K) = \max\{l : \pi_K = \{K_1, K_2, ..., K_l, \{i_1\}, ..., \{i_r\}\}\}, K \quad N,$$

where $K_1 \quad \dots \quad K_l \quad \{i_1\} \quad \dots \quad \{i_r\} = K$.

**Definition 4.** *In the cooperative game with the characteristic function $v^G(K)$, the core is the set of imputations*

$$C = \{x : \sum_{i \ N} x_i = v^G(N), \quad \sum_{i \ S} x_i \geq v^G(S), \quad S \quad N\}.$$

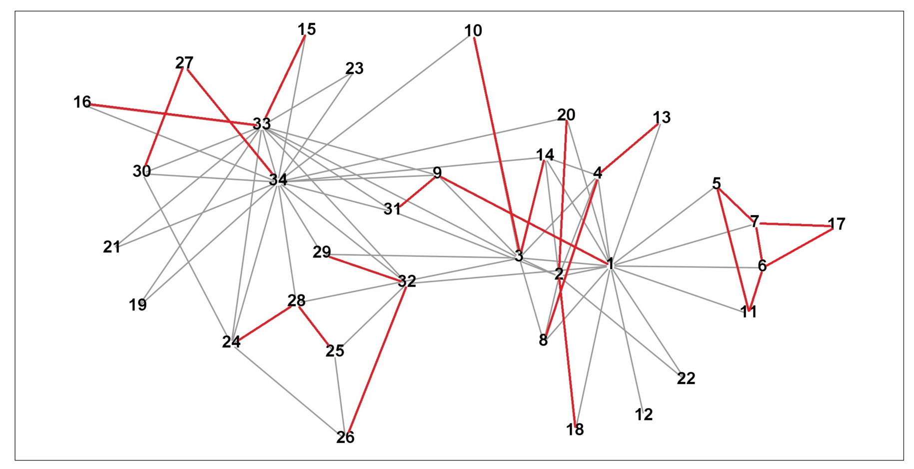
Consider the linear programming problem

$$\min \sum_{i=1}^{n} x_i$$

$$\text{s.t.} \sum_{j=1}^{d+1} x_{i_j} \geq 1, \quad L_k \quad L;$$

$$x_i \geq 0, \quad i \quad N.$$

**Theorem 4 (Dotsenko, Mazalov 2021).** The graph packing game with simple paths is balanced if and only if the dual linear programming problem has an integer optimal solution. The core of the balanced graph packing game is the solution of the primal linear problem.




(d) Packing by triples for Zachary's karate club network.

## Main Publications


- Mazalov, V., Trukhina, L. (2014). Generating functions and the Myerson vector in comm. networks. *Disc. Math. Appl.*, 24(5).
- Avrachenkov, K., Kondratev A., Mazalov, V., Rubanov, D. (2018). Network partitioning algorithm as cooperative games. *Comput. Soc. Net.*, 5(11).