Dynamical systems in ecology and economics

Aleksandr Kirillov, Aleksandra Ivanova, Aleksandr Sazonov

Institute of Applied Mathematical Research

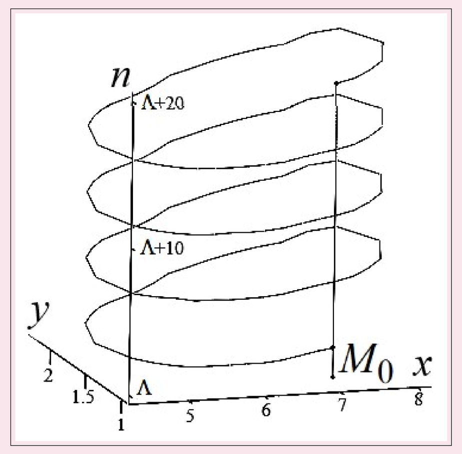
Karelian Research Centre of the Russian Academy of Sciences

Periodic control processes in the problem of biocommunity species composition preservation

A dynamical model describing several stages of interaction between the populations of predators and prey in a patch is proposed. The transition from one stage to another is determined by the values of the function of the trophic attractiveness of a species patch which has the form $n(t) = \Lambda + \int_0^t y(\tau)\left(\frac{x(\tau)}{y(\tau)} - \lambda\right) d\tau$, where $0 < \lambda, \Lambda = n(0)$ are the given constant thresholds, x, y are the prey and predator population sizes, respectively. If $n(t) > \Lambda$, then the patch is attractive and interactios between the populations are described by the Lotka-Volterra system with taking specimens out

$$\begin{aligned} \dot{x} &= x(a - by - u_1), \\ \dot{y} &= y(kbx - m - u_2), \\ \dot{n} &= x - \lambda y, \end{aligned} \tag{1}$$

where a is the prey population increase rate, b is the prey consumption rate per predator, k is the part of energy derived from prey biomass that the predator population uses for reproduction, $k \in (0,1)$, m is the predator mortality rate, $u_1 \in [0, a), u_2 \geqslant 0$ are the taking specimens out intensities of prey and predators, respectively. If $n(t) < \Lambda$, then the patch is unattractive and the predator population enters the migration stage. If $n = \Lambda$, then the behavior of population is determined by the sign of $x - \lambda y$. The model also describes the process of colonization of a patch by predators. The problem of preserving the species composition by taking specimens out is solved. The taking specimens out intensities are considered as a control functions. The control which permits coordinating antropogenic impact with the natural development of a biocommunity is constructed. As a result, the control process acquires the periodic or quasiperiodic nature, which depends on the initial population sizes.



(a) The phase trajectory of (1) corresponding the constructed periodic control process.

The proposed model is described by a three-dimensional system of ordinary differential equations with variable structure. The qualitative analysis of the system is carried out.

Main Publications

- Kirillov, A., Ivanova, A. (2017). Equilibrium and control in the biocommunity species composition preservation problem. *Automation and Remote Control*, 78(8).
- Kirillov, A., Ivanova, A. (2021). Numerical modeling of a periodic process that preserves the species structure of a biocommunity. *Mathematical Models and Computer Simulations*, 33(6).

Dynamics of endogenous technology development

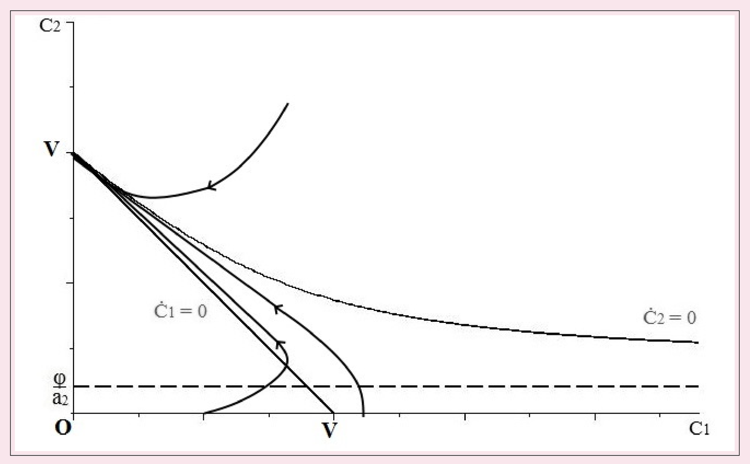
The mathematical modeling of the economic evolution enables to describe the processes more precisely than empirical studies and allow for the prediction of the behaviour of economic systems which is extremely important for the development of the economic growth policies.

$$\begin{aligned} \dot{C}_1 &= \tfrac{1-\varphi_1}{\lambda_1} C_1 (V - C_1 - \ldots - C_N), \\ \dot{C}_i &= \tfrac{1-\varphi_i}{\lambda_i} C_i (V - C_1 - \ldots - C_N) + \varphi_{i-1} C_{i-1}, \end{aligned} \tag{2}$$

where $i = 2, \ldots, N$, C_i is the capital at the i-th level, φ_i is the share of the i-th level capital for developing of the $(i+1)$-th level, λ_i is the profit per capacity unit at the i-th level, V is the economic niche volume.

The mathematical model of sector capital distribution dynamics over efficiency levels with shared economic niche in the form of a system of nonlinear differential equations is proposed. The authors update the approach of Polterovich-Henkin by taking into account the boundedness of economic growth due to the limitation of markets, resources and other factors. For this purpose, the concept of economic niche volume is introduced. The economic niche volume is a limit integrated capital value, for which the growth rate is so low that there is no capital growth. The models enable the prediction of the dynamical behavior of the economic system. The qualitative analysis of these models is presented. The equilibria of the constructed dynamical models are determined, their global stability is proved. The global stability of the equilibrium $P = (0, \ldots, 0, V)$ means that the capital at the highest level approaches the value of the economic niche volume, while the capitals at other levels tend to zero. Thus, the highest level beats the competition in the case of a shared economic niche.

Note that the method for the proof of global stability was developed by the authors. Besides, the Jacobi matrix at the point $P = (0, \ldots, 0, V)$ has all the eigenvalues equal to zero, except one negative, which does not allow to establish the local stability using linear approximation.



(b) The global stability of the equilibrium of (2).

Main Publications

- Kirillov, A., Sazonov, A. (2019). Global Schumpeterian dynamics with structural variations. *Bulletin of the South Ural State University*, 12(3). Series "Mathematical modelling, programming & computer software".
- Kirillov, A., Sazonov, A. (2020). The global stability of the Schumpeterian dynamical system. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 16(4).

Stabilization of the biological wastewater treatment process

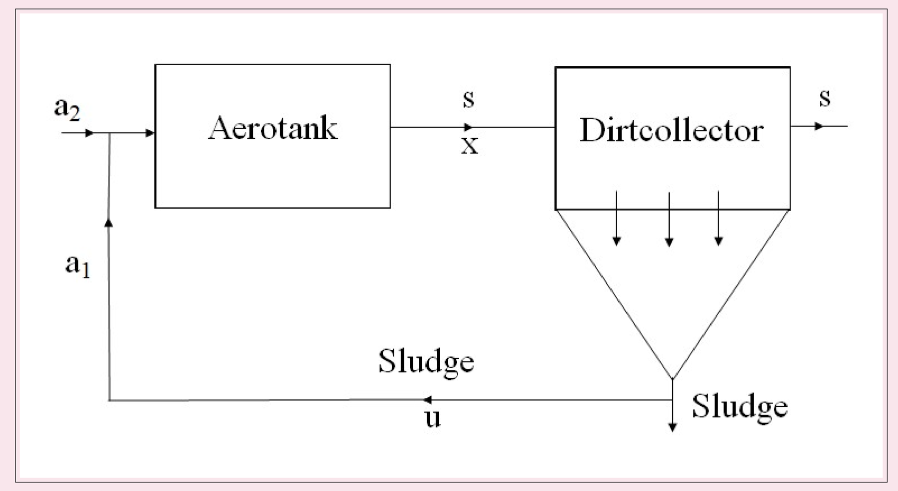
$$\begin{aligned} \dot{x} &= Q(a_1, u) + f(x, s) - (b + u)x, \\ \dot{s} &= R(a_2, b) - \tfrac{1}{Y} f(x, s) - (b + u)s, \end{aligned} \tag{3}$$

where s is substrate concentration, x are microorganisms, u is control, $u \in [0, \bar{u}]$, Y is coefficient of conversion of the pollutant substrate into the biomass of microorganisms, $b, a1$ are the substrate input speed and concentration, $u, a2$ are the rate and concentration of biomass in the return flow, $Y, b, a1, a2$ are positive constants, $Q(a1, u)$, $R(a2, b)$ are the rate of consumption of microorganisms and substrates, respectively, at the aerotank entrance, $f(x, s) \geq 0$ is the rate of microorganism biomass growth due to substrate oxidation. Functions Q, R, f ensure the uniqueness of the solution to the Cauchy problem, $Q \in [0, \bar{Q}]$, $R \in [0, \bar{R}]$.

It is natural to assume that $f(x, 0) = f(0, s) = 0$ and $f(x, s) \leq cx$, where $c \geq 0$ is a constant. The latter limitation reflects the saturation inherent in the process of substrate oxidation by microorganisms. It is easy to show that the positive quarter $R_+^2 = \{(x, s) : x \geq 0, s \geq 0\}$ is an invariant set of system (3). Also, it follows from the form of the right-hand sides of system (3) and the restrictions on Q, R, f, which were introduced above, that for sufficiently large constants $\bar{x} \geq 0$, $\bar{s} \geq 0$, the invariant set of system (3) is a rectangle P of the form: $P = \{(x, s) : x \in [0, \bar{x}], s \in [0, \bar{s}]\}$.

A mathematical model of the biological wastewater treatment process based on simple balance equations is proposed. The model of the process in the form of nonlinear system of ordinary differential equations has a general nature: the functions describing oxidation and the input flow are not specified. The invariant set is found and its asymptotic global stability is proved. On the basis of the proposed model, a technique has been developed to stabilize the wastewater treatment process when the pollutant inflow is non-stationary, piecewise-constant.

The constructive method of normal local stabilization of nonlinear autonomous systems, applicable to ecological processes control, is developed.



(c) The biological wastewater treatment process.

Main Publications

- Kirillov, A. (2019). The method of normal local stabilization. *Issues of Analysis*, 8(26), No 1.
- Kirillov, A., Danilova, I. (2020). Dynamics of the biological wastewater treatment process under variable pollution input flow: invariant sets and stabilization. *Transactions of the Karelian Research Centre of the Russian Academy of Sciences*, No 7. Mathematical modeling and information technologies series.