Configuration graphs and forest fire models

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Introduction

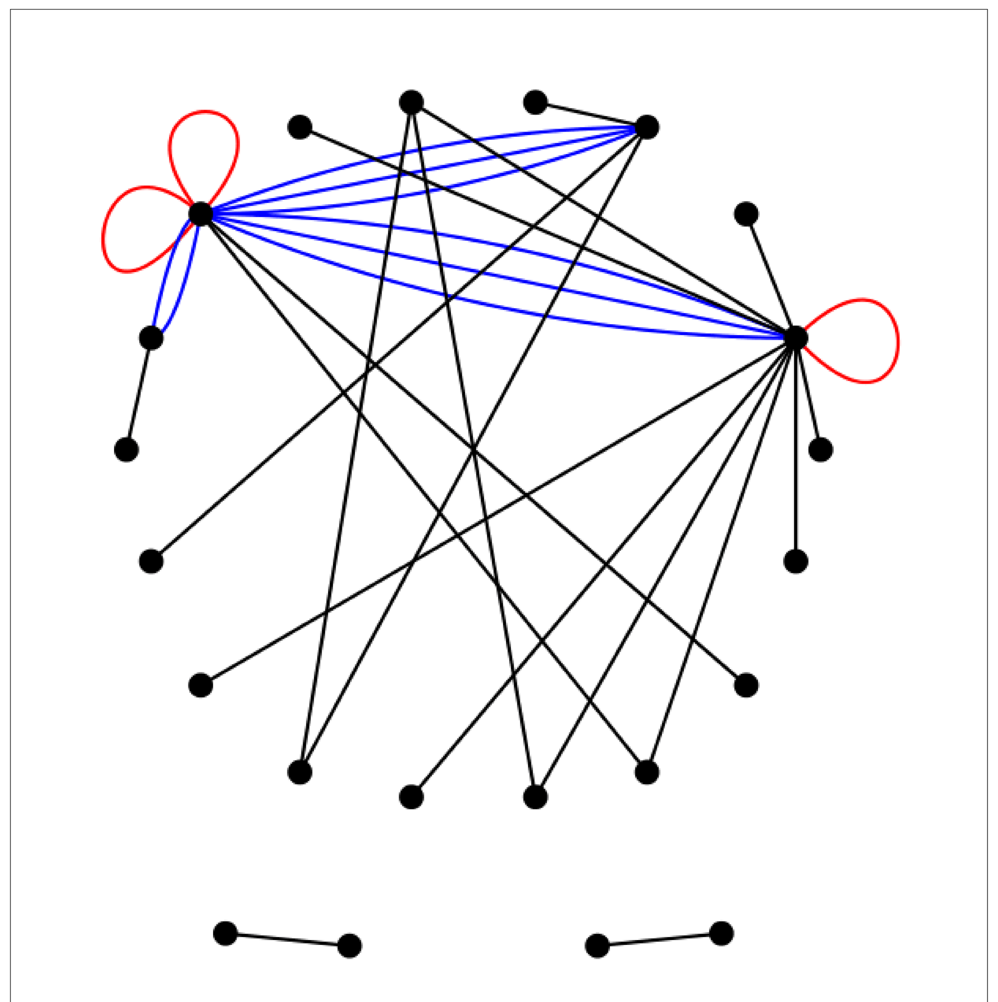
The study of random graphs has been gaining interest in the past decades due to the wide use of these models for the description of massive data networks. Such models can be used for representing transport, telephone and electricity networks, social relationships, telecommunications and, of course, the main global network – Internet. While considering these networks it has been noted that their topology could be described by random graphs, with vertex degrees being independent and identically distributed random variables. Let ξ be a random variable equal to the degree of any vertex. In a number of papers, the authors showed that in modeling of huge networks it is more appropriate to use the following degree distribution of the random variable ξ:

$$\mathbf{P}\{\xi = k\} = k^{-\tau} - (k+1)^{-\tau}, \tag{1}$$

where $k = 1, 2, \ldots,\ \tau > 0$. One of the most commonly used types of network models are the configuration graph. Lately in the theory of random graphs there appeared new directions of research connected with transmission of various destructive influences through graph links. In particular we consider such a graph where the destructive process is interpreted as a fire spread over the graph and they could be used for modeling forest fires as well as banking system defaults. We took attempts for constructing such models based on configuration random graphs.

Power-law configuration graphs

For graph construction each vertex is given a certain degree in accordance with the degree distribution (1). Vertex degrees form stubs (or semiedges) that are numbered in an arbitrary order. The graph is constructed by joining all the stubs pairwise equiprobably to form edges. If the sum of vertex degrees is odd one stub is added to a random vertex. Obviously such graphs may have loops and multiple edges.



(a) An example of configuration graph

Forest fires in random environment

Degree structure

We consider power-law configuration graphs with vertex degrees following the distribution (1) under the condition that the parameter τ is a random variable uniformly distributed on the interval $[a, b]$, where $0 < a < b < \infty$. Then, the random variable ξ has the following distribution:

$$\begin{aligned} p_1 &= \mathbf{P}\{\xi = 1\} = 1 - \frac{1}{(b-a)\ln 2}\left(\frac{1}{2^a} - \frac{1}{2^b}\right), \\ p_k &= \mathbf{P}\{\xi = k\} = \frac{1}{(b-a)\ln k}\left(\frac{1}{k^a} - \frac{1}{k^b}\right) - \frac{1}{(b-a)\ln(k+1)}\left(\frac{1}{(k+1)^a} - \frac{1}{(k+1)^b}\right), \end{aligned} \tag{2}$$

where $k = 2, 3, \ldots$.

We denote by $\xi_{(N)}$ and μ_r the random variables equal to the maximum vertex degree and the number of vertices with degree r, respectively. If vertex degrees follow the distribution (1), $\xi_{(N)}$ is proportional to $N^{1/\tau}$ as $N \to \infty$, and μ_r for large r is proportional to $Nr^{-(\tau+1)}$. Let us consider the limit behaviour of these characteristics in the case of vertex degree distribution (2).

Theorem 1. *Let $N \to \infty$. Then for any fixed x*

$$\mathbf{P}\{a \ln \xi_{(N)} - \ln N + \ln\ln N + \ln(b-a) - \ln a \le x\} \to e^{-e^{-x}}.$$

Theorem 2. Let $N \to \infty$ and k is a natural number. The following assertions are true.

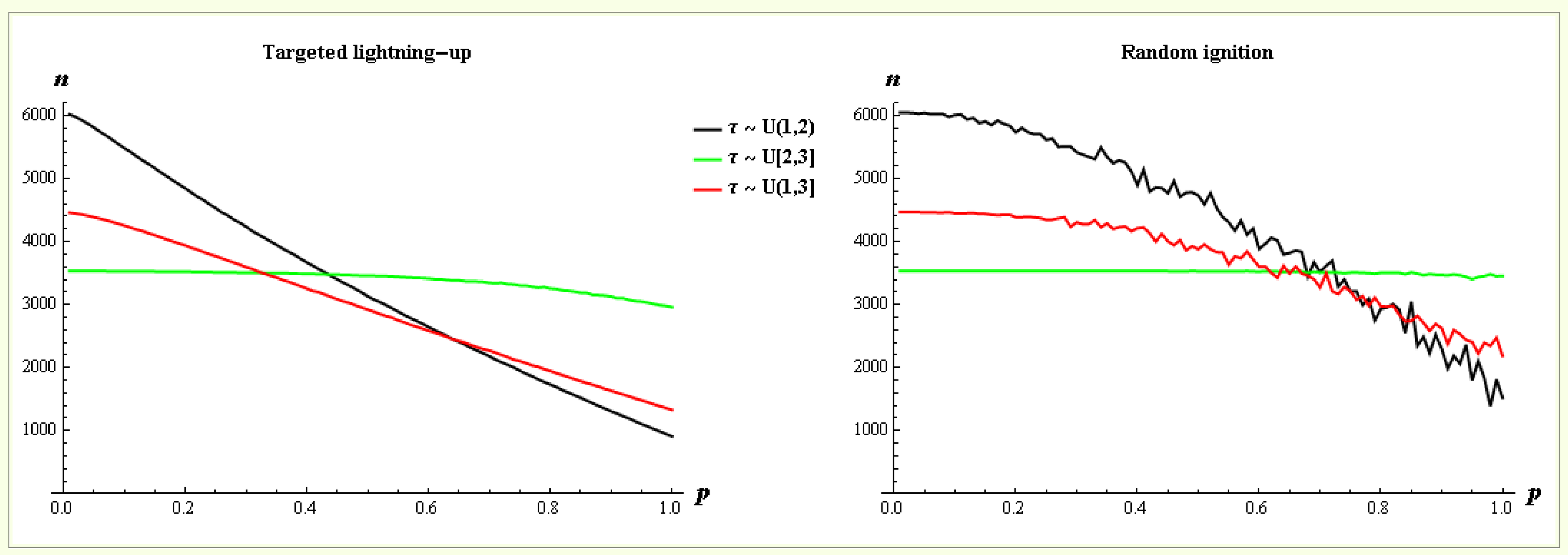
1. If $Np_r \to \infty$ then uniformly in k such that $u_r = (k - Np_r)/\sqrt{Np_r(1-p_r)}$ lies in any fixed finite interval
$$\mathbf{P}\{\mu_r = k\} = (2\pi Np_r(1-p_r))^{-1/2} e^{-u_r^2/2}(1 + o(1)).$$
2. If $r \to \infty$ then uniformly in k such that $(k - Np_r)/\sqrt{Np_r}$ lies in any fixed finite interval
$$\mathbf{P}\{\mu_r = k\} = \frac{(Np_r)^k}{k!} e^{-Np_r}(1 + o(1)).$$

Distribution (2), as well as (1), is the same for all vertices, thus it is not yet a random environment. It is more natural to suppose that vertex degrees are defined by Equation (1), where the value of τ is chosen from the interval $[a, b]$ equiprobably for each vertex. Comparison of the results showed their similarity, which means that the study of the graphs' behaviour in the considered random environment can be replaced by the study of the model with an averaged vertex degree distribution (2).

Forest fires

Simulations were held for the three intervals $[a, b]$: $(1, 2)$, $[2, 3]$, $(1, 3]$ with the N values obtained from (3) being equal to 6046, 3536 and 4470, respectively. There were analysed two cases of starting a fire propagation process: targeted lightning-up of a vertex with the highest degree and random ignition of an equiprobably chosen vertex. When the fire starts it spreads through the incident edges to connected vertices with the probability of fire transition p. This probability could be either a predefined value $p \in (0, 1]$ fixed for all the graph edges or a random variable following the standard uniform distribution. Both cases are studied. The purpose is to find the optimal interval $[a, b]$ of the distribution (2) that would ensure maximum survival of graph vertices in case of a fire.

For all the three intervals relations between the average number of vertices surviving in a fire n and the probability of fire transition p were found. The following plots show how the number of remaining vertices n depends on the probability p in the two fire-start cases:



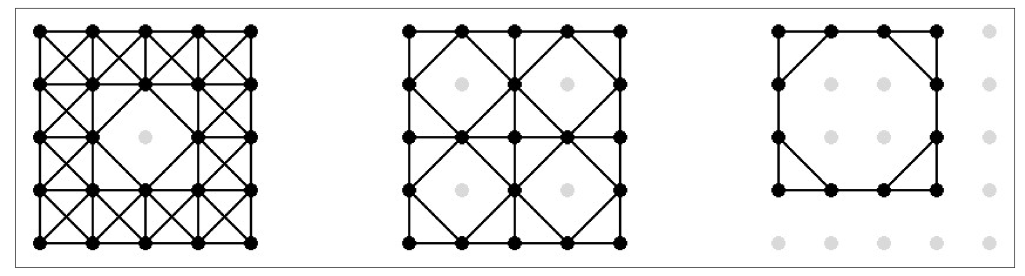
(b) Relation between the number of surviving vertices n and the probability of fire transition p.

The results showed that graphs with the distribution parameter $\tau \in (1, 2)$ are more resilient in cases of lower values of the probability of fire transition p for both fire ignitions. Thus, as the value of p increases the topology with the parameter $\tau \in [2, 3]$ will ensure a better survival of nodes.

Forest fire model

Let us consider graph vertices as trees growing in a certain area of a real forest, and graph edges as possible pathways of transferring a fire from one tree to another. Such an approach had been called a "forest fire model".

The study of a forest fire model imposes some constraints on the graphs being considered. Since graph vertices are viewed as trees growing on a limited area of a real forest, their number as well as the number of vertices in a corresponding graph has to be limited. We propose to use an auxiliary square lattice graph sized 100×100. Two graph vertices are connected if on the corresponding tree topology a fire can move from one tree to another.



(c) Lattice graph topology for an average vertex degree 7, 5.33 and 2.66, respectively.

We consider 15 different relative allocations of edges and vertices on lattice graphs. Let m be an averaged inner vertex degree. For a fully packed square lattice $m = 8$. For each lattice graph topology corresponding values of m and N are calculated. Let the size of power-law configuration graph be equal to the size of the auxiliary lattice graph. Knowing that $m = \zeta(\tau)$ (where $\zeta(x)$ is the Riemann zeta function), we obtained a regression relation between the power-law configuration graph size $N \le 10000$ and the parameter τ of vertex degree distribution (1):

$$N = [9256\tau^{-1.05}], \qquad R^2 = 0.97. \tag{3}$$

It is clear that if initially there are not many trees in the area, there will not be many trees left after the fire. On the other hand, if there are too many trees and they grow too close to each other, then after the fire there will not be many of them left either. That is why there appears the problem of finding the optimal value of the parameter τ of vertex degree distribution (1) that would ensure maximum survival of trees in case of a fire.

Main Publications

- Leri, M., Pavlov, Yu. (2014). Power-law random graphs' robustness: Link saving and forest fire model. *Austrian J. Stat.*, 43(4).
- Leri, M., Pavlov, Yu. (2016). Forest fire models on configuration random graphs. *Fundamenta Informaticae*, 145(3).
- Leri, M., Pavlov, Yu. (2017). Random graphs' robustness in random environment. *Austrian J. Stat.*, 46(3-4).