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A FISHERY GAME MODEL WITH MIGRATION: RESERVED TERRITORY APPROACH

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1. Introduction

A dynamic game model of a bioresource management problem (fisheries) is considered. The center (state) which determines the reserved portion of the reservoir (where fishing is prohibited), and the players (fishing farms) which harvest fish are the participants of the game. Each player is an independent decision maker, guided by the considerations of maximizing the profit from fish sale. We consider finite and infinite planning horizon. Pontryagin's maximal principle and Hamilton-Jacobi-Bellman equation were applied to determine Nash and Stakelberg equlibriums.

2. Game model

Let us consider the center, which determines the reserved area of the reservoir denoted by s, $0 \le s \le 1$. We consider the strategies of the two players which exploit the fish stock during T time periods. Let us divide the water area into two parts: S_1 and S_2, where fishing is prohibited and allowed, respectively. Denote by x_1 and x_2 the size of the population per unit area of S_1 and S_2, respectively. Then $s = S_1/S$ is the reserved area. There is a migratory exchange between the two parts of the reservoir with the exchange coefficient $\gamma = q/s$, where q is the exchange rate.

The dynamics of the fishery is described by the system of equations:

$$\begin{cases} x_1'(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t)) \,, \\ x_2'(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - u(t) - v(t) \,, \ x_i(0) = x_i^0 \,, \end{cases} \tag{1}$$

where $x_1(t) \ge 0$ – size of the population at time t in the reserved area; $x_2(t) \ge 0$ – size of the population at time t in the area where fishing is allowed; ε – natural growth rate of the population; $u(t) \ge 0$ – first farm's fishing efforts at time t; $v(t) \ge 0$ – second farm's fishing efforts at time t; $s(t)$ – reserved portion of the reservoir and $\gamma_i = q/s$ – coefficients of the migratory exchange, $i = 1, 2$.

Then the payoffs of the two players over a fixed time period $[0, T]$ are

$$\begin{aligned} J_1 &= \int_0^T e^{-rt}[m_1((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_1 u(t)^2 - p_1 u(t)]\, dt \,, \\ J_2 &= \int_0^T e^{-rt}[m_2((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_2 v(t)^2 - p_2 v(t)]\, dt \,, \end{aligned} \tag{2}$$

where $\bar{x}_i(t)$ – size of the population which is optimal for reproduction, m_i – penalty for deviation from the state $(\bar{x}_1, \bar{x}_2)$, c_i – catching costs of the i-th player and p_i – market price for each player, $i = 1, 2$.

Let's denote

$$c_{ir} = c_i e^{-rt}\ ,\ m_{ir} = m_i e^{-rt}\ ,\ p_{ir} = c_i e^{-rt}\ ,\ i = 1, 2.$$

We consider different optimality principles.

1.1. Nash optimal solution

We are interested in the optimal solution of the following problem:

$$\begin{cases} J_1(u^*, v^*) \le J_1(u, v^*)\ ,\ \forall u\ , \\ J_1(u^*, v^*) \le J_1(u^*, v)\ ,\ \forall v\ . \end{cases}$$

The Hamilton function for player I is

$$\begin{aligned} H_1 &= m_{1r}((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_{1r}u(t)^2 - p_{1r}u(t) + \\ &+ \lambda_{11}(t)(\varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t))) + \\ &+ \lambda_{12}(t)(\varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - u(t) - v(t))\ . \end{aligned}$$

Let's find the maximum of H_1:

$$\frac{\partial H_1}{\partial u} = 2c_{1r}u(t) - p_{1r} - \lambda_{12}(t) = 0\ .$$

Then the maximum is achieved at the point

$$u(t) = \frac{\lambda_{12}(t) + p_{1r}}{2c_{1r}}\ .$$

According to the maximum principle [4]

$$\begin{aligned} \lambda'_{11}(t) &= -\tfrac{\partial H_1}{\partial x_1} = -2m_{1r}(x_1(t) - \bar{x}_1) - \lambda_{11}(t)(\varepsilon - \gamma_1) - \lambda_{12}(t)\gamma_2\ , \\ \lambda'_{12}(t) &= -\tfrac{\partial H_1}{\partial x_2} = -2m_{1r}(x_2(t) - \bar{x}_2) - \lambda_{12}(t)(\varepsilon - \gamma_2) - \lambda_{11}(t)\gamma_1\ , \end{aligned}$$

and the transversability conditions are

$$\lambda_{1i}(T) = 0\ ,\ i = 1, 2\ .$$

In a similar manner, for player II

$$\begin{aligned} H_2 &= m_{2r}((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_{2r}v(t)^2 - p_{2r}v(t) + \\ &+ \lambda_{21}(t)(\varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t))) + \\ &+ \lambda_{22}(t)(\varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - u(t) - v(t))\ . \end{aligned}$$

It yields

$$v(t) = \frac{\lambda_{22}(t) + p_{2r}}{2c_{2r}}\ ,$$

and equations for additional variables are

$$\begin{aligned} \lambda'_{21}(t) &= -\tfrac{\partial H_2}{\partial x_1} = -2m_{2r}(x_1(t) - \bar{x}_1) - \lambda_{21}(t)(\varepsilon - \gamma_1) - \lambda_{22}(t)\gamma_2\ , \\ \lambda'_{22}(t) &= -\tfrac{\partial H_2}{\partial x_2} = -2m_{2r}(x_2(t) - \bar{x}_2) - \lambda_{22}(t)(\varepsilon - \gamma_2) - \lambda_{21}(t)\gamma_1\ , \end{aligned}$$

with the transversability conditions

$$\lambda_{2i}(T) = 0\,,\ i = 1,2\,.$$

Finally, to find the Nash equilibrium we have to solve the following system:

$$\begin{cases} x_1'(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t))\,, \\ x_2'(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - \frac{\lambda_{12}(t)+p_{1r}}{2c_{r1}} - \frac{\lambda_{22}(t)+p_{2r}}{2c_{2r}}\,, \\ \lambda_{11}'(t) = -2m_{1r}(x_1(t) - \bar{x}_1) - \lambda_{11}(t)(\varepsilon - \gamma_1) - \lambda_{12}(t)\gamma_2\,, \\ \lambda_{12}'(t) = -2m_{1r}(x_2(t) - \bar{x}_2) - \lambda_{12}(t)(\varepsilon - \gamma_2) - \lambda_{11}(t)\gamma_1\,, \\ \lambda_{21}'(t) = -2m_{2r}(x_1(t) - \bar{x}_1) - \lambda_{21}(t)(\varepsilon - \gamma_1) - \lambda_{22}(t)\gamma_2\,, \\ \lambda_{22}'(t) = -2m_{2r}(x_2(t) - \bar{x}_2) - \lambda_{22}(t)(\varepsilon - \gamma_2) - \lambda_{21}(t)\gamma_1\,, \\ \lambda_{i1}(T) = \lambda_{i2}(T) = 0\,,\ x_i(0) = x_i^0\,. \end{cases}$$

Let's recall our notations and introduce new variables:

$$\bar{\lambda}ij = \lambda ij e^{rt}\,,\ i,j = 1,2\,.$$

Then for these variables we get the system:

$$\begin{cases} x_1'(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t))\,, \\ x_2'(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - \frac{\bar{\lambda}_{12}(t)+p_1}{2c_1} - \frac{\bar{\lambda}_{22}(t)+p_2}{2c_2}\,, \\ \bar{\lambda}_{11}'(t) = -2m_1(x_1(t) - \bar{x}_1) - \bar{\lambda}_{11}(t)(\varepsilon - \gamma_1 - r) - \bar{\lambda}_{12}(t)\gamma_2\,, \\ \bar{\lambda}_{12}'(t) = -2m_1(x_2(t) - \bar{x}_2) - \bar{\lambda}_{12}(t)(\varepsilon - \gamma_2 - r) - \bar{\lambda}_{11}(t)\gamma_1\,, \\ \bar{\lambda}_{21}'(t) = -2m_2(x_1(t) - \bar{x}_1) - \bar{\lambda}_{21}(t)(\varepsilon - \gamma_1 - r) - \bar{\lambda}_{22}(t)\gamma_2\,, \\ \bar{\lambda}_{22}'(t) = -2m_2(x_2(t) - \bar{x}_2) - \bar{\lambda}_{22}(t)(\varepsilon - \gamma_2 - r) - \bar{\lambda}_{21}(t)\gamma_1\,, \\ \bar{\lambda}_{i1}(T) = \bar{\lambda}_{i2}(T) = 0\,,\ x_i(0) = x_i^0\,. \end{cases} \tag{3}$$

Theorem 1.

$$u^*(t) = \frac{\bar{\lambda}_{12}(t) + p_1}{2c_1}$$

and

$$v^*(t) = \frac{\bar{\lambda}_{22}(t) + p_2}{2c_2}\,,$$

with $\bar{\lambda}_{i2}$, $i = 1,2$ *satisfing (3), form the Nash optimal solution of the problem (1)-(2).*

Proof. See [8].

Example.

Modelling was carried out for the following values: $q = 0.2$, $\gamma_1 = \gamma_2 = q/s$, $\varepsilon = 0.08$, $m_1 = m_2 = 0.09$, $c_1 = c_2 = 10$, $p_1 = p_2 = 100$, $T = 200$, $r = 0.1$.

Let the initial size of the population be $x_1(0) = 50$, $x_2(0) = 50$. And the optimal for reproduction population sizes are $\bar{x}_1 = 100$ and $\bar{x}_2 = 100$.

You can see the optimal values of $u^*(t)$ and $v^*(t)$ (equal for both players) in Fig.3. The figure indicates that the player's strategies should equal 8 almost all the time. The size of the population in the reserved area grows from 50 to 160 individuals (Fig.1). The size of the population in the area where fishing is allowed grows from 50 to 140 individuals (Fig.2).

The players' profits given that they use the optimal strategies, are $J_1 = J_2 = 103.6457652$.

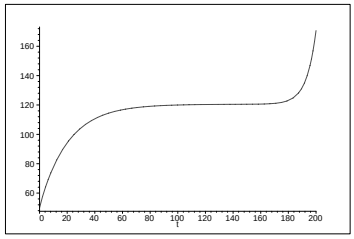

Figure 1. Values of $x_1^*(t)$

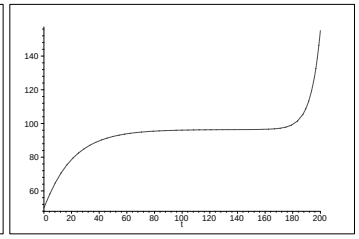

Figure 2. Values of $x_2^*(t)$

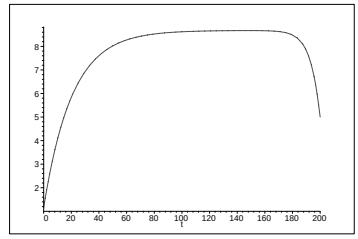

Figure 3. Values of $u^*(t)$

1.2. Stackelberg optimal solution

We are interested in the optimal solution of the following problem:

$$\begin{cases} \max\limits_{v\in R(u^*)} J_1(u^*,v) \le \max\limits_{v\in R(u)} J_1(u,v)\,, \\ \text{where } R(u)=\{v \mid J_2(u,v)\le J_2(u,v')\}\,, \\ \text{and } v^*\in R(u^*)\,. \end{cases}$$

For solving this problem we use Pontryagin's maximum principle modified for a two level game.

We fix some strategy $u(t)$, and the Hamilton function for player II is

$$\begin{aligned} H_2 &= m_{2r}((x_1(t)-\bar{x}_1)^2+(x_2(t)-\bar{x}_2)^2)+c_{2r}v(t)^2-p_{2r}v(t)+ \\ &+ \lambda_{21}(t)(\varepsilon x_1(t)+\gamma_1(x_2(t)-x_1(t)))+ \\ &+ \lambda_{22}(t)(\varepsilon x_2(t)+\gamma_2(x_1(t)-x_2(t))-u(t)-v(t))\,. \end{aligned}$$

Wherefore

$$v(t)=\frac{\lambda_{22}(t)+p_{2r}}{2c_{2r}}\,,$$

and the equations for additional variables are

$$\begin{aligned} \lambda_{21}'(t) &= -\tfrac{\partial H_2}{\partial x_1} = -2m_{2r}(x_1(t)-\bar{x}_1)-\lambda_{21}(t)(\varepsilon-\gamma_1)-\lambda_{22}(t)\gamma_2\,, \\ \lambda_{22}'(t) &= -\tfrac{\partial H_2}{\partial x_2} = -2m_{2r}(x_2(t)-\bar{x}_2)-\lambda_{22}(t)(\varepsilon-\gamma_2)-\lambda_{21}(t)\gamma_1\,, \end{aligned}$$

with the transversability conditions

$$\lambda_{2i}(T)=0\,,\ i=1,2\,.$$

We substitute this strategy of player II into the system (1) and combine it with the equations for additional variables

$$\begin{cases} x_1'(t)=\varepsilon x_1(t)+\gamma_1(x_2(t)-x_1(t))\,, \\ x_2'(t)=\varepsilon x_2(t)+\gamma_2(x_1(t)-x_2(t))-u-\frac{\lambda_{22}(t)+p_{2r}}{2c_{2r}}\,, \\ \lambda_{21}'(t)=-2m_{2r}(x_1(t)-\bar{x}_1)-\lambda_{21}(t)(\varepsilon-\gamma_1)-\lambda_{22}(t)\gamma_2\,, \\ \lambda_{22}'(t)=-2m_{2r}(x_2(t)-\bar{x}_2)-\lambda_{22}(t)(\varepsilon-\gamma_2)-\lambda_{21}(t)\gamma_1\,, \\ \lambda_{21}(T)=\lambda_{22}(T)=0\,,\ x_i(0)=x_i^0\,. \end{cases}$$

Then we repeat all operations using the Pontryagin's maximum principle. The Hamilton function for player I is

$$\begin{aligned} H_1 &= m_{1r}((x_1(t)-\bar{x}_1)^2+(x_2(t)-\bar{x}_2)^2)+c_{1r}u(t)^2-p_{1r}u(t)+ \\ &+ \lambda_{11}(t)(\varepsilon x_1(t)+\gamma_1(x_2(t)-x_1(t)))+ \\ &+ \lambda_{12}(t)(\varepsilon x_2(t)+\gamma_2(x_1(t)-x_2(t))-u(t)-\tfrac{\lambda_{22}(t)+p_{2r}}{2c_{2r}})+ \\ &+ \mu_1(t)(-2m_{2r}(x_1(t)-\bar{x}_1)-\lambda_{21}(t)(\varepsilon-\gamma_1)-\lambda_{22}(t)\gamma_2)+ \\ &+ \mu_2(t)(-2m_{2r}(x_2(t)-\bar{x}_2)-\lambda_{22}(t)(\varepsilon-\gamma_2)-\lambda_{21}(t)\gamma_1) \end{aligned}$$

Let's find the maximum of H_1:

$$\frac{\partial H_1}{\partial u}=2c_{1r}u(t)-p_{1r}-\lambda_{12}(t)=0\,.$$

Then the maximum is achieved at the point

$$u(t)=\frac{\lambda_{12}(t)+p_{1r}}{2c_{1r}}\,.$$

According to the maximal principle [4]

$$\begin{cases} \lambda_{11}'(t)=-2m_{1r}(x_1(t)-\bar{x}_1)-\lambda_{11}(t)(\varepsilon-\gamma_1)-\lambda_{12}(t)\gamma_2+2m_{2r}\mu_1(t)\,, \\ \lambda_{12}'(t)=-2m_{1r}(x_2(t)-\bar{x}_2)-\lambda_{12}(t)(\varepsilon-\gamma_2)-\lambda_{11}(t)\gamma_1+2m_{2r}\mu_2(t)\,, \\ \mu_1'(t)=-\frac{\partial H_1}{\partial\lambda_{21}}=\mu_1(t)(\varepsilon-\gamma_1)+\mu_2(t)\gamma_1\,, \\ \mu_2'(t)=-\frac{\partial H_1}{\partial\lambda_{22}}=\frac{\lambda_{12}(t)}{2c_{2r}}+\mu_2(t)(\varepsilon-\gamma_2)+\mu_1(t)\gamma_2\,, \end{cases}$$

and the transversability conditions are

$$\lambda_{2i}(T)=0\,,\ \mu_i(0)=0\,.$$

Finally, to find the Stackelberg equilibrium we have to solve the following system:

$$\begin{cases} x_1'(t)=\varepsilon x_1(t)+\gamma_1(x_2(t)-x_1(t))\,, \\ x_2'(t)=\varepsilon x_2(t)+\gamma_2(x_1(t)-x_2(t))-\frac{\lambda_{12}(t)+p_{1r}}{2c_{1r}}-\frac{\lambda_{22}(t)+p_{2r}}{2c_{2r}}\,, \\ \lambda_{11}'(t)=-2m_{1r}(x_1(t)-\bar{x}_1)-\lambda_{11}(t)(\varepsilon-\gamma_1)-\lambda_{12}(t)\gamma_2+2m_{2r}\mu_1(t)\,, \\ \lambda_{12}'(t)=-2m_{1r}(x_2(t)-\bar{x}_2)-\lambda_{12}(t)(\varepsilon-\gamma_2)-\lambda_{11}(t)\gamma_1+2m_{2r}\mu_2(t)\,, \\ \lambda_{21}'(t)=-2m_{2r}(x_1(t)-\bar{x}_1)-\lambda_{21}(t)(\varepsilon-\gamma_1)-\lambda_{22}(t)\gamma_2\,, \\ \lambda_{22}'(t)=-2m_{2r}(x_2(t)-\bar{x}_2)-\lambda_{22}(t)(\varepsilon-\gamma_2)-\lambda_{21}(t)\gamma_1\,, \\ \mu_1'(t)=\mu_1(t)(\varepsilon-\gamma_1)+\mu_2(t)\gamma_1\,, \\ \mu_2'(t)=\frac{\lambda_{12}(t)}{2c_{2r}}+\mu_2(t)(\varepsilon-\gamma_2)+\mu_1(t)\gamma_2\,, \\ \lambda_{i1}(T)=\lambda_{i2}(T)=0\,,\ x_i(0)=x_i^0\,,\ \mu_i(0)=0\,. \end{cases}$$

Let's recall our notations and introduce new variables:

$$\bar{\lambda}ij=\lambda ije^{rt}\,,$$

Then for these variables we get the system:

$$\begin{cases} x_1'(t)=\varepsilon x_1(t)+\gamma_1(x_2(t)-x_1(t))\,, \\ x_2'(t)=\varepsilon x_2(t)+\gamma_2(x_1(t)-x_2(t))-\frac{\bar{\lambda}_{12}(t)+p_1}{2c_1}-\frac{\bar{\lambda}_{22}(t)+p_2}{2c_2}\,, \\ \bar{\lambda}_{11}'(t)=-2m_1(x_1(t)-\bar{x}_1)-\bar{\lambda}_{11}(t)(\varepsilon-\gamma_1-r)-\bar{\lambda}_{12}(t)\gamma_2+2m_2\mu_1(t)\,, \\ \bar{\lambda}_{12}'(t)=-2m_1(x_2(t)-\bar{x}_2)-\bar{\lambda}_{12}(t)(\varepsilon-\gamma_2-r)-\bar{\lambda}_{11}(t)\gamma_1+2m_2\mu_2(t)\,, \\ \bar{\lambda}_{21}'(t)=-2m_2(x_1(t)-\bar{x}_1)-\bar{\lambda}_{21}(t)(\varepsilon-\gamma_1-r)-\bar{\lambda}_{22}(t)\gamma_2\,, \\ \bar{\lambda}_{22}'(t)=-2m_2(x_2(t)-\bar{x}_2)-\bar{\lambda}_{22}(t)(\varepsilon-\gamma_2-r)-\bar{\lambda}_{21}(t)\gamma_1\,, \\ \mu_1'(t)=\mu_1(t)(\varepsilon-\gamma_1)+\mu_2(t)\gamma_1\,, \\ \mu_2'(t)=\frac{\bar{\lambda}_{12}(t)}{2c_2}+\mu_2(t)(\varepsilon-\gamma_2)+\mu_1(t)\gamma_2\,, \\ \bar{\lambda}_{i1}(T)=\bar{\lambda}_{i2}(T)=0\,,\ x_i(0)=x_i^0\,,\ \mu_i(0)=0\,. \end{cases} \tag{4}$$

Theorem 2. *The strategies*

$$u^*(t) = \frac{\bar{\lambda}_{12}(t) + p_1}{2c_1}$$

and

$$v^*(t) = \frac{\bar{\lambda}_{22}(t) + p_2}{2c_2},$$

with $\bar{\lambda}_{i2}$, $i = 1, 2$ *satisfing (4), form the Stackelberg optimal solution of the problem (1)-(2).*

Example.

Modelling was carried out for the same values as in section 1.1.

You can see the optimal values of $u^*(t)$ in Fig.6 and those of $v^*(t)$ in Fig.7. The figures indicate that player's I startegy should equal 6 almost all the time, and player's II strategy – equal 12. The size of the population in the reserved area grows from 50 to 200 individuals (Fig.4). The size of the population in the area where fishing is allowed grows from 50 to 180 individuals (Fig.5),

The players' profits given that they use the optimal strategies, are $J_1 = -253.7144427$, $J_2 = 943.6131678$.

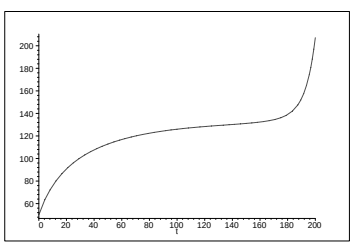

Figure 4. Values of $x_1^*(t)$

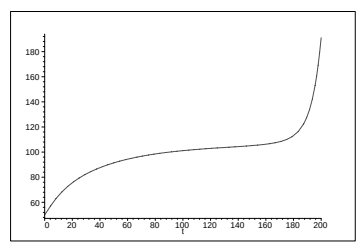

Figure 5. Values of $x_2^*(t)$

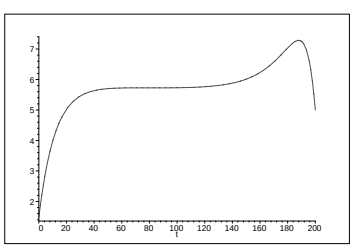

Figure 6. Values of $u^*(t)$

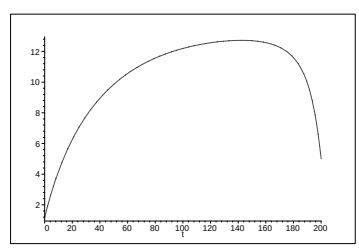

Figure 7. Values of $v^*(t)$

Let's compare the players' payoffs when we use different optimality principles.

Profit of player I, which corresponds to different sizes of the reserved area $s(t)$, are shown in Table 1, and profit of player II – in Table 2.

Table 1. Player I profit

$s(t)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Nash	11591	6670	3177	1017	103	353	1689	4040	7341
Stakelberg	11240	6329	2837	673	-253	-26	1277	3582	6820

Table 2. Player II profit

$s(t)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Nash	11591	6670	3177	1017	103	353	1689	4040	7341
Stakelberg	12426	7498	4002	1846	943	1210	2571	4955	8294

In the case of the Nash equilibrium both players are in the same conditions, so their strategies and profits are equal.

In the case of the Stackelberg equilibrium player I is the leader and, as Tables 1, 2 show, this equilibrium is better for player I, but worse for player II. This solution gives the gain to player I, whereas player II carries all the expenses of maintaining a stable population development.

2. Model over an infinite horizon

The dynamics of the fishery, as before, is described by the system of equations:

$$\begin{cases} x_1'(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t)) \,, \\ x_2'(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - u(t) - v(t) \,, \;\; x_i(0) = x_i^0 \,, \end{cases} \tag{5}$$

where $x_1(t) \geq 0$ – size of the population at time t in the reserved area; $x_2(t) \geq 0$ – size of the population at time t in the area where fishing is allowed; ε – natural growth rate of the population; $u(t) \geq 0$ – first farm's fishing efforts at time t; $v(t) \geq 0$ – second farm's fishing efforts at time t; $s(t)$ – reserved portion of the reservoir and $\gamma_i = q/s$ – coefficients of the migratory exchange, $i = 1, 2$.

Then the discounted payoffs of the two players over an infinite horizon at a rate r are

$$\begin{aligned} J_1 &= \textstyle\int_0^\infty e^{-rt}[m_1((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_1 u(t)^2 - p_1 u(t)]\, dt \,, \\ J_2 &= \textstyle\int_0^\infty e^{-rt}[m_2((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_2 v(t)^2 - p_2 v(t)]\, dt \,, \end{aligned} \tag{6}$$

where $\bar{x}_i(t)$ – population size optimal for reproduction, m_i – penalty for deviation from the state $(\bar{x}_1, \bar{x}_2)$, c_i – catching costs of the i-th player and p_i – market price for each player, $i = 1, 2$.

When we investigated the model with a finite planning horizon, we used Pontryagin's maximum principle. The infinite horizon gives us an opportunity to use Hamilton–Jacobi–Bellman equation.

Let's consider different optimality principles for infinite horizon model.

2.1. Nash optimal solution

Let's fix the second player's strategy and consider the problem of determining the optimal strategy of player I.

Define the value function $V(x)$ for our problem

$$V(x_1, x_2) = \min_u \{ \int_0^\infty e^{-rt}[m_1((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_1 u(t)^2 - p_1 u(t)]\} \,.$$

The Hamilton–Jacobi–Bellman equation is

$$\begin{array}{rcl} rV(x_1,x_2) & = & \min\limits_{u}\{m_1((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2)+c_1u^2-p_1u+\frac{\partial V}{\partial x_1}(\varepsilon x_1+\gamma_1(x_2-x_1))+ \\ & + & \frac{\partial V}{\partial x_2}(\varepsilon x_2+\gamma_2(x_1-x_2)-u-v)\} \end{array}$$

Let's find the maximum over u:

$$2c_1u-\frac{\partial V}{\partial x_2}-p_1=0\,.$$

Then the maximum is achieved at the point

$$u=(\frac{\partial V}{\partial x_2}+p_1)/2c_1\,.$$

Substituting it into the equation, we get

$$\begin{array}{rcl} rV(x_1,x_2)=m_1((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2)-\frac{(\frac{\partial V}{\partial x_2}+p_1)^2}{4c_1} & + & \frac{\partial V}{\partial x_1}(\varepsilon x_1+\gamma_1(x_2-x_1))+ \\ & + & \frac{\partial V}{\partial x_2}(\varepsilon x_2+\gamma_2(x_1-x_2)-v) \end{array}$$

It is easy to verify that the quadratic form provides a solution to this equation.
Let $V(x_1,x_2)=a_1x_1^2+b_1x_1+a_2x_2^2+b_2x_2+kx_1x_2+l$.
Then the player's I strategy is

$$u(x)=\frac{2a_2x_2+b_2+kx_1+p_1}{2c_1}\,,$$

where the coefficients satisfy the system of equations

$$\left\{\begin{array}{rcl} ra_1 & = & m_1-\frac{k^2}{4c_1}+2a_1(\varepsilon-\gamma_1)+k\gamma_2 \\ rb_1 & = & -2m_1\bar{x}_1-\frac{kb_2}{2c_1}-\frac{kp_1}{2c_1}+b_1(\varepsilon-\gamma_1)+b_2\gamma_2-kv \\ ra_2 & = & m1-\frac{a_2^2}{c_1}+2a_2(\varepsilon-\gamma_2)+k\gamma_1 \\ rb_2 & = & -2m_1\bar{x}_2-\frac{a_2b_2}{c_1}-\frac{a_2p_1}{c_1}+b_2(\varepsilon-\gamma_2)+b_1\gamma_1-2a_2v \\ rk & = & -\frac{a_2k}{c_1}+k(\varepsilon-\gamma_1)+2a_1\gamma_1+2a_2\gamma_2+k(\varepsilon-\gamma_2) \\ rl & = & m_1\bar{x}_1^2+m_1\bar{x}_2^2-\frac{b_2^2}{4c_1}-\frac{b_2p_1}{2c_1}-\frac{p_1^2}{4c_1}-b_2v \end{array}\right. \tag{7}$$

Analogously for player II

$$v(x)=\frac{2\alpha_2x_2+\beta_2+k_2x_1+p_2}{2c_2}\,,$$

where the coefficients satisfy the system of equations

$$\left\{\begin{array}{rcl} r\alpha_1 & = & m_2-\frac{k_2^2}{4c_1}+2\alpha_1(\varepsilon-\gamma_1)+k_2\gamma_2 \\ r\beta_1 & = & -2m_2\bar{x}_1-\frac{k_2\beta_2}{2c_1}-\frac{k_2p_1}{2c_1}+\beta_1(\varepsilon-\gamma_1)+\beta_2\gamma_2-k_2u \\ r\alpha_2 & = & m2-\frac{\alpha_2^2}{c_1}+2\alpha_2(\varepsilon-\gamma_2)+k_2\gamma_1 \\ r\beta_2 & = & -2m_2\bar{x}_2-\frac{\alpha_2\beta_2}{c_1}-\frac{\alpha_2p_1}{c_1}+\beta_2(\varepsilon-\gamma_2)+\beta_1\gamma_1-2\alpha_2u \\ rk_2 & = & -\frac{\alpha_2k_2}{c_1}+k_2(\varepsilon-\gamma_1)+2\alpha_1\gamma_1+2\alpha_2\gamma_2+k_2(\varepsilon-\gamma_2) \\ rl_2 & = & m_2\bar{x}_1^2+m_2\bar{x}_2^2-\frac{\beta_2^2}{4c_1}-\frac{\beta_2p_1}{2c_1}-\frac{p_1^2}{4c_1}-\beta_2u \end{array}\right. \tag{8}$$

So, we proved the following theorem.

Theorem 3.

$$u^*(x) = \frac{2a_2x_2 + b_2 + kx_1 + p_1}{2c_1}$$

and

$$v^*(x) = \frac{2\alpha_2x_2 + \beta_2 + k_2x_1 + p_2}{2c_2},$$

form the Nash optimal solution of the problem (5)-(6), where the coefficients are defined from (7) and (8).

Example.

Modelling was carried out for the following values: $q = 0.2$, $\gamma_1 = \gamma_2 = q/s$, $\varepsilon = 0.08$, $m_1 = m_2 = 0.09$, $c_1 = c_2 = 10$, $p_1 = p_2 = 100$, $T = 200$, $r = 0.1$.

Let the initial size of the population be $x_1(0) = 50$, $x_2(0) = 50$. And the optimal for reproduction population sizes are $\bar{x}_1 = 100$ and $\bar{x}_2 = 100$.

You can see the optimal values of $u^*(t)$ and $v^*(t)$ (equal for both players) in Fig.10. The figure indicates that players' strategies should increase from 2 to 8. The size of the population in the reserved area grows from 50 to 120 individuals (Fig.8). The size of the population in the area where fishing is allowed grows from 50 to 90 individuals (Fig.9).

The players' payoffs given that they use the optimal strategies, are $J_1 = J_2 = 388.0627019$.

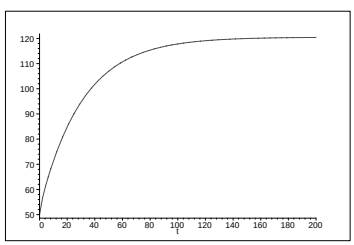

Figure 8. Values of $x_1^*(t)$

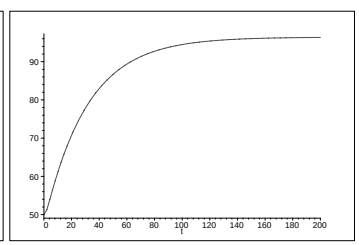

Figure 9. Values of $x_2^*(t)$

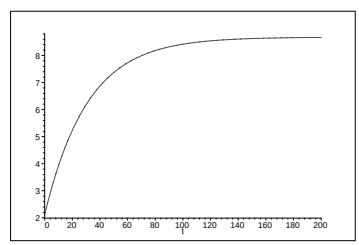

Figure 10. Values of $u^*(t)$

2.2. Stackelberg optimal solution

Using Hamilton–Jacobi–Bellman equation for player II we get

$$v(x) = \frac{2\alpha_2x_2 + \beta_2 + k_2x_1 + p_2 + \sigma u}{2c_2},$$

where the coefficients are defined from (8).

Define the value function $V(x)$ for player's I problem

$$V(x_1, x_2) = \min_u \{\int_0^\infty e^{-rt}[m_1((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_1u(t)^2 - p_1u(t)]\}.$$

The Hamilton–Jacobi–Bellman equation is

$$\begin{aligned} rV(x_1, x_2) &= \min_u\{m_1((x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2) + c_1u^2 - p_1u + \tfrac{\partial V}{\partial x_1}(\varepsilon x_1 + \gamma_1(x_2 - x_1)) + \\ &+ \tfrac{\partial V}{\partial x_2}(\varepsilon x_2 + \gamma_2(x_1 - x_2) - u - \tfrac{2\alpha_2x_2 + \beta_2 + k_2x_1 + p_2 + \sigma u}{2c_2})\} \end{aligned}$$

Let's find the maximum over u:

$$2c_1u - \frac{\partial V}{\partial x_2}(1+\frac{\sigma}{2c_2}) - p_1 = 0\,,$$

wherefore

$$u = (\frac{\partial V}{\partial x_2}(2c_2+\sigma) + 2p_1c_2)/4c_1c_2\,.$$

Substituting it into the equation, we get

$$\begin{array}{rcl} rV(x_1,x_2) & = & m_1((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2) + (\frac{\partial V}{\partial x_2})^2\frac{(2c_2+\sigma)^2}{8c_1c_2^2} + \frac{p_1^2}{2c_1} + \frac{\partial V}{\partial x_1}(\varepsilon x_1 + \gamma_1(x_2-x_1)) + \\ & + & \frac{\partial V}{\partial x_2}(x_1(\gamma_2 - \frac{k_2}{2c_2}) + x_2(\varepsilon - \gamma_2 - \frac{\alpha_2}{c_2}) + \frac{2c_2+\sigma}{2c_1c_2} - \frac{\beta_2+p_2}{2c_2}) \end{array}$$

It is easy to verify that the quadratic form provides a solution to this equation.

Let $V(x_1,x_2) = a_1x_1^2 + b_1x_1 + a_2x_2^2 + b_2x_2 + gx_1x_2 + l$.

Then the player's I strategy is

$$u(x) = ((2a_2x_2 + b_2 + gx_1)(2c_2+\sigma) + 2p_1c_2)/4c_1c_2\,,$$

where the coefficients satisfy the system of equations

$$\left\{\begin{array}{rcl} ra_1 & = & m_1 + g^2\frac{(2c_2+\sigma)^2}{8c_1c_2^2} + 2a_1(\varepsilon-\gamma_1) + g(\gamma_2 - \frac{k_2}{2c_2}) \\ rb_1 & = & -2m_1\bar{x}_1 + 2b_2g\frac{(2c_2+\sigma)^2}{8c_1c_2^2} + g\frac{(2c_2+\sigma)}{2c_1c_2}b_1(\varepsilon-\gamma_1) + b_2(\gamma_2 - \frac{k_2}{2c_2}) - \frac{g(\beta_2+p_2)}{2c_2} \\ ra_2 & = & m1 + a_2\frac{(2c_2+\sigma)^2}{2c_1c_2^2} + g\gamma_1 + 2a_2(\varepsilon - \gamma_2 - \frac{\alpha_2}{c_2}) \\ rb_2 & = & -2m_1\bar{x}_2 + a_2b_2\frac{(2c_2+\sigma)^2}{2c_1c_2^2} + a_2\frac{(2c_2+\sigma)}{c_1c_2} + b_2(\varepsilon-\gamma_2-\frac{\alpha_2}{c_2}) + b_1\gamma_1 - a_2\frac{\beta_2+p_2}{c_2} \\ rg & = & a_2g\frac{(2c_2+\sigma)^2}{2c_1c_2^2} + g(\varepsilon-\gamma_1) + 2a_1\gamma_1 + 2a_2(\gamma_2 - \frac{k_2}{2c_2}) + g(\varepsilon-\gamma_2-\frac{\alpha_2}{c_2}) \\ rl & = & m_1\bar{x}_1^2 + m_1\bar{x}_2^2 + b_2^2\frac{(2c_2+\sigma)^2}{8c_1c_2^2} + b_2\frac{(2c_2+\sigma)}{2c_1c_2} + \frac{p_1^2}{2c_1} - \frac{b_2(\beta_2+p_2)}{2c_2} \end{array}\right. \quad (9)$$

So, we proved the following theorem.

Theorem 4. *The strategies*

$$u^*(x) = ((2a_2x_2 + b_2 + gx_1)(2c_2+\sigma) + 2p_1c_2)/4c_1c_2$$

and

$$v^*(x) = (\sigma(2a_2x_2+b_2+gx_1)(2c_2+\sigma) + 4c_1c_2(2\alpha_2x_2+\beta_2+k_2x_1) + 2c_2(\sigma p_1 + 2c_1p_2))/8c_1c_2^2\,,$$

form the Stackelberg optimal solution of the problem (5)-(6), where the coefficients are defined from (8) and (9).

Example.

Modelling was carried out for the same values as in section 2.1.

You can see the optimal values of $u^*(t)$ in Fig.13 and those of $v^*(t)$ in Fig.14. The figure indicates that player's I strategy should increase from 4.6 to 5.2, and player's II strategy – from 0 to 14. The size of the population in the reserved area grows from 50 to 120 individuals (Fig.11). The size of the population in the area where fishing is allowed grows from 50 to 100 individuals (Fig.12).

The players' profits given that they use the optimal strategies, are $J_1 = 163.4798825$ and $J_2 = 1153.667808$.

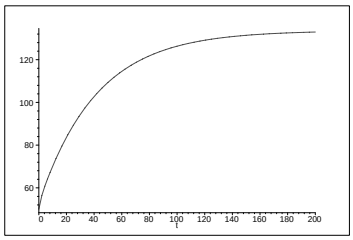

Figure 11. Values of $x_1^*(t)$

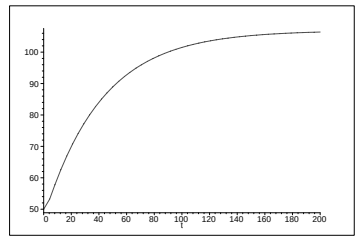

Figure 12. Values of $x_2^*(t)$

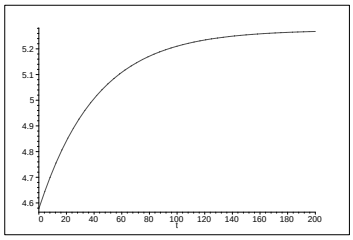

Figure 13. Values of $u^*(t)$

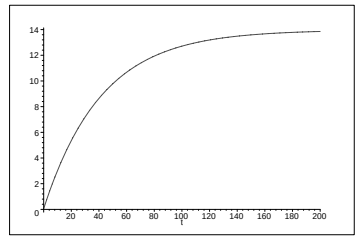

Figure 14. Values of $v^*(t)$

Let's compare the players' profits when we use different optimal principls.

Profit of player I, which corresponds to different sizes of the reserved area $s(t)$, are shown in Table 3, and profit of player II – in Table 4.

Table 3. Player I profit

$s(t)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Nash	11868	6930	3431	1280	388	668	2037	4414	7720
Stakelberg	11641	6702	3203	1053	163	446	1818	4199	7508

Table 4. Player II profit

$s(t)$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Nash	11868	6930	3431	1280	388	668	2037	4414	7720
Stakelberg	12642	7706	4207	2052	1153	1425	2784	5149	8442

As in the case of the finite planning horizon, the Stackelberg equilibrium is better for player I, but worse for player II than the Nash equilibrium.

In practice, the Hamilton–Jacobi–Bellman equation is more convenient for long–term planning.

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