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A FISHERY GAME MODEL WITH AGE–DISTRIBUTED POPULATION: RESERVED TERRITORY APPROACH

Vladimir V. Mazalov
Anna N. Rettieva
Institute of Applied Mathematical Research
Karelian Research Center of RAS
Russia

1. Introduction

A dynamic game model of a bioresource management problem (fisheries) is considered. The center (state), which determines the reserved portion of the reservoir (where fishing is prohibited), and the players (fishing firms), which harvest fish, are the participants of this game. Each player is an independent decision maker, guided by the considerations of maximizing the profit from fish sale. In the traditional statement (see Clark, 1985, Ehtamo and Hamalainen, 1993, Hamalainen, Kaitala and Haurie, 1984, Haurie and Tolwinski, 1984, Tolwinski, Haurie and Leitmann, 1986) the center's objective is catch regulation by introduction of quotas. In this paper the center's task is to determine the optimal reserved portion to maintain stable population development in the reservoir in the long term and the harvest level, sufficient for demand satisfaction.

2. Game model for one player

Let us consider the center, which determines the reserved area of the reservoir, denoted by s, $0 \leq s \leq 1$. The area in which fishing is allowed, is equal to $1 - s$, respectively. We consider the player's strategy, which exploits the fish stock during T time periods.

The dynamics of the fishery is described by the equation:

$$x'(t) = F(x(t)) - qE(t)(1 - s(t))x(t),\ 0 \leq t \leq T,\ x(0) = x_0\,, \tag{1}$$

where $x(t) \geq 0$ – size of the population at a time t; F – natural growth function of the population; $E(t) \geq 0$ – firm's fishing efforts measured as the number of vessels involved in fishing at time t, $s(t)$ – reserved portion of the reservoir, and $q > 0$ – catchability coefficient related to the unit fishing effort of the firm.

Assume that population evolves in accordance with Ferhulst model of the form:

$$F(x) = rx(1 - x/K)\,,$$

where r – the intrinsic growth rate, and K – maximal natural object capacity.

Then the net revenue over a fixed time period $[0,T]$ is

$$\begin{aligned} J &= g(x(T)) + \\ &+ \int_0^T e^{-\rho t}[\Pi(q,s(t),x(t),E(t)) \cdot qE(t)(1-s(t))x(t) - c^0 E(t)]dt\,, \end{aligned}$$

where ρ – discount rate, c^0 – catching costs for one vessel and Π – price function, specified as

$$\Pi(q,s(t),x(t),E(t)) = p - kqE(t)(1-s(t))x(t),\ p,k>0\,.$$

Function $g(x)$ describes the salvage value of the stock at time T and

$$g'(x) \geq 0,\ g''(x) \leq 0\,.$$

Let's rewrite the payoff function in the following way

$$\begin{aligned} J &= g(x(T)) + \\ &+ \int_0^T [-\tfrac{1}{2}aE(t)^2(1-s(t))^2 x(t)^2 + bE(t)(1-s(t))x(t) - cE(t)]dt\,, \end{aligned}$$

where

$$\begin{aligned} a &= 2kq^2 \exp(-\rho t)\,, \\ b &= pq \exp(-\rho t)\,, \\ c &= c^0 \exp(-\rho t)\,. \end{aligned}$$

We are interested in the optimal solution of the following problem:

$$\begin{cases} \max(J(E(t)))\,, \\ \text{where } x(t) \text{ is defined in (1)}\,. \end{cases} \tag{2}$$

Theorem 1. *Let $E^*(t)$, $x^*(t)$, $\lambda(t)$ satisfy the equation (1) and*

$$E^*(t) = \frac{(b-q\lambda(t))(1-s(t))x^*(t) - c}{a(1-s(t))^2 {x^*(t)}^2},\ 0 \leq t \leq T\,,$$

$$\begin{aligned} \lambda'(t) &= -E^*(t)(1-s(t))(b - aE^*(t)(1-s(t))x^*(t)) - \\ &- \lambda(t)(F'(x^*(t)) - qE^*(t)(1-s(t)))\,,\ 0 \leq t \leq T\,, \end{aligned}$$

$$\lambda(T) = g'_x(x^*(T))\,,$$

and let

$$x^*(t) > x_m = \frac{2c^0}{pq(1-s(t))}\,,$$

$$b - \lambda(t)q \geq \frac{3c}{2x_m(1-s(t))}\,.$$

Then $E^*(t)$ is the solution of the problem (2).

Proof. Using the maximal principle, the Hamilton function is

$$\begin{array}{rcl} H(x,E,\lambda,s,t) & = & -\frac{1}{2}aE^2(1-s)^2x^2+bE(1-s)x-cE+ \\ & + & \lambda(F(x)-qE(1-s)x)\,. \end{array}$$

Let's find the maximum of H.

$$\frac{\partial H}{\partial E}=-aE(1-s)^2x^2+b(1-s)x-c-\lambda q(1-s)x=0\,.$$

Then the maximum is achieved at the point

$$E(x,\lambda,t)=\frac{(b-q\lambda)(1-s)x-c}{a(1-s)^2x^2}\,.$$

According to the maximal principle [6]

$$\lambda'=-\frac{\partial H}{\partial x}=-E(1-s)(b-aE(1-s)x)-\lambda(F'_x-qE(1-s))\,,$$

and the transversability condition is

$$\lambda(T)=g'_x(x^*(T))\,.$$

Substituting the optimal $E^*(t)$ to (1) and the equation for $\lambda(t)$ we obtain the following system of equations:

$$\left\{\begin{array}{rcl} x'(t) & = & rx(t)(1-\frac{x(t)}{K})-\frac{q}{a}(b-\frac{c}{x(t)(1-s(t))})+\lambda(t)\frac{q^2}{a} \\ \lambda'(t) & = & -\frac{c}{ax^2(t)(1-s(t))}(b-\frac{c}{x(t)(1-s(t))})- \\ & - & \lambda(t)\cdot(r-\frac{2rx(t)}{K}-\frac{cq}{ax^2(t)(1-s(t))}) \\ x(0) & = & x_0 \\ \lambda(T) & = & g'_x(x^*(T)) \end{array}\right. \tag{3}$$

Let's prove the optimality of the solution. At first we show, that $\lambda(t)>0$, $\forall t$. Represent the second equation in (3) as

$$\lambda'(t)=-L(t)-M(t)\lambda(t)\,,$$

where $L(t)>0$ $\forall t$, because $x^*(t)>\frac{c}{b(1-s(t))}=\frac{c^0}{pq(1-s(t))}$, and $x^*(t)>x_m$.
Consider the equation for $\lambda(t)$.

1. The solution of the equation

$$\lambda'(t)=-M(t)\lambda(t)\,.$$

is

$$\lambda(t)=c_0(t)e^{-\int\limits_0^t M(\tau)d\tau}\,.$$

2. $c_0(t)$ satisfies the equation

$$c_0'(t)e^{-\int\limits_0^t M(\tau)d\tau} = -L(t)\,,$$

from which

$$c_0 = -\int\limits_0^t L(\tau)e^{\int\limits_0^\tau M(p)dp}\,d\tau + c_1\,.$$

Then

$$\lambda(t) = e^{-\int\limits_0^t M(\tau)d\tau}(c_1 - \int\limits_0^t L(\tau)e^{\int\limits_0^\tau M(p)dp})\,d\tau\,.$$

From the transversability condition

$$\lambda(T) > 0\,,$$

then

$$c_1 \geq \int\limits_0^T L(\tau)e^{\int\limits_0^\tau M(p)dp}\,d\tau \geq \int\limits_0^t L(\tau)e^{\int\limits_0^\tau M(p)dp}\,d\tau\ ,\forall t \in [0,T]\,.$$

We conclude that

$$\lambda(t) > 0\ \ ,\forall t\,.$$

Define

$$H^0(x,\lambda,t,s) = \max_{E\in R} H(x,E,\lambda,t,s) = H(x,E^*,\lambda,t,s)\,.$$

Substituting E^*, we obtain

$$H^0(x,\lambda,t,s) = \frac{c^2}{2a(1-s)^2x^2} - \frac{c(b-\lambda q)}{a(1-s)x} + \frac{(b-\lambda q)^2}{2a} + \lambda F(x)\,.$$

The inequality $b - \lambda q \geq \frac{3c}{2x(1-s)}$,$\forall x > x_m$ yields

$$H^0_{xx}(x,\lambda,t,s) = \frac{2c}{ax^3(1-s)}(\frac{3c}{2x(1-s)} - (b-\lambda q)) - \frac{2r}{K}\lambda(t) \leq 0\,.$$

Therefore H^0 is concave.
Let $E^*(t), x^*(t), \lambda(t)$ be as in the theorem's conditions.
By definition

$$H(x,E,\lambda,t,s) \leq H^0(x,\lambda,t,s)\,,$$

from concavity

$$H^0(x,\lambda,t,s) \leq H^0(x^*,\lambda,t,s) + H^0_x(x^*,\lambda,t,s)(x(t)-x^*(t))\,.$$

The condition $\lambda'(t) = -H^0_x(x^*,\lambda,t,s)$ gives

$$H(x,E,\lambda,t,s) \leq H^0(x^*,\lambda,t,s) - \lambda'(t)(x-x^*)\,.$$

Integrating it from 0 to T we obtain

$$\int_0^T H(x,E,\lambda,t,s)dt \le \int_0^T H^0(x^*,\lambda,t,s)dt - \lambda(t)(x-x^*)|_0^T + \int_0^T \lambda(t)(x'(t)-x^*(t)')dt\,,$$

$$J + \int_0^T \lambda(t)(F(x)-qE(1-s)x)\,dt - g(x(T)) \le J^* + \int_0^T \lambda(t)(F(x^*)-qE^*(1-s)x^*)\,dt -$$
$$-g(x^*(T)) - \lambda(T)(x(T)-x^*(T)) + \int_0^T \lambda(t)x'(t)dt - \int_0^T \lambda(t)x^*(t)'dt\,,$$

$$J + \int_0^T \lambda(t)x'(t)dt - g(x(T)) \le J^* + \int_0^T \lambda(t)x^*(t)'dt - g(x^*(T)) -$$
$$-\lambda(T)(x(T)-x^*(T)) + \int_0^T \lambda(t)x'(t)dt - \int_0^T \lambda(t)x^*(t)'dt\,.$$

Then

$$J - (g(x(T)) - g(x^*(T))) \le J^* - \lambda(T)(x(T)-x^*(T))\,,$$

$$\begin{aligned} J &\le J^* - \lambda(T)(x(T)-x^*(T)) + (g(x(T)) - g(x^*(T))) \le \\ &\le J^* - \lambda(T)(x(T)-x^*(T)) + g'_x(x^*(T))(x(T)-x^*(T)) = \\ &= J^* - \lambda(T)(x(T)-x^*(T)) + \lambda(T))(x(T)-x^*(T))\,. \end{aligned}$$

Finally,

$$J \le J^*\,.$$

Consequently, E^* is the solution of the problem. □

So, if the center's strategy $s(t)$, $t \in [0,T]$ is known to the player, he can find the optimal behaviour $E^*(t)$. This, in turn, becomes known to the center, which, depending on the player's strategy, receives either a gain or a loss.

We examine the following functionals determining the center's gain:

1. $I_1 = -\int_0^T (x(t)-\bar{x}(t))^2 dt\,,$

   where $\bar{x}(t)$ – size of the population which is optimal for reproduction.

   In this case I_1 is the center's cost for population's regeneration.

2. $I_2 = -\int_0^T (U(t)-\hat{x}(t))^2 dt\,,$

3. $I_3 = -\int_0^T |U(t)-\hat{x}(t)| \cdot \alpha \cdot \Pi(q,s(t),x(t),E(t))dt\,,$

   where $U(t) = qE(t)(1-s(t))x(t)$ – player's catch at time t, $\hat{x}(t)$ – consumption level determined by demand, $\Pi(q,s(t),x(t),E(t)) = p - kqE(t)(1-s(t))x(t),\ p,k > 0\,.$

The center's profit is determined as the cost of demand satisfaction expressed in case 3) in monetary units, taking into account the transport cost.

In Section 4 the values of J, I_1, I_2 and I_3 are found for various $s(t)$ and Nash bargaining solution is obtained.

3. Game models for the population with three age classes

3.1. Game model for a naturally regenerating population

Let's consider a model, which allows the existence of more that one age class of fish in the reservoir. In this subsection, let it be three classes, namely – young fish, middle–age fish and old fish. Fish of two adult classes are able of reproduction. The model is given for the naturally regenerating population. The player can catch either middle–age fish or old fish using different fishing nets.

Let us consider the center, which determines the reserved area of the reservoir, denoted by s, $0 \leq s \leq 1$. The portion where fishing is allowed, is thus equal to $1 - s$. We consider the player's strategy, which exploits the fish stock during T time periods.

The dynamics of the fishery is described by the system of equations:

$$\begin{cases} x_1'(t) = k(l_2\sigma_2 x_2(t) + l_3\sigma_3 x_3(t)) - (\alpha_1 + \beta_1)x_1(t) - dx_1^2(t)\,, \\ x_2'(t) = \alpha_1 x_1(t) - (\beta_2 + \alpha_2)x_2(t) - q_2(t)E(t)(1 - s(t))x_2(t)\,, \\ x_3'(t) = \alpha_2 x_2(t) - \beta_3 x_3(t) - q_3(t)E(t)(1 - s(t))x_3(t),\ 0 \leq t \leq T,\ x_i(0) = x_i^0\,, \end{cases} \tag{4}$$

where $x_1(t) \geq 0$ – number of young fish at time t, $x_2(t) \geq 0$ – number of middle–age fish at time t, $x_3(t) \geq 0$ – number of old fish at time t, l_2, l_3 – escapement rates, σ_2, σ_3 – mean fertility in the second and third age classes, k – egg survival rate, α_1, α_2 – recruitment rates, $\beta_1 + dx_1(t), \beta_2, \beta_3$ – mortalities, $E(t) \geq 0$ – firm's fishing efforts measured as the number of vessels involved in fishing at time t, $q_2(t)$, $q_3(t) > 0$, $q_2(t) + q_3(t) = 0.02$ – catchability coefficients for middle–age fish and old fish related to the unit fishing effort of the firm (fishing net's ratio for the two classes), $s(t)$ – proportion of the reserved area in the reservoir.

The player's revenue is

$$\begin{aligned} J \;&=\; g(x_2(T), x_3(T)) + \int_0^T e^{-\rho t}[\Pi_2(q_2(t), s(t), x_2(t), E(t)) \cdot q_2(t)E(t)(1 - s(t))x_2(t) + \\ &+\; \Pi_3(q_3(t), s(t), x_3(t), E(t)) \cdot q_3(t)E(t)(1 - s(t))x_3(t) - c^0 E(t)(q_2(t) + q_3(t))]dt\,, \end{aligned}$$

where ρ – discount rate, c^0 – catching cost for one vessel and Π_i, $i = 2, 3$ – price functions for different age classes, specified as

$$\Pi_2(q_2(t), s(t), x_2(t), E(t)) = p_2 - kq_2(t)E(t)(1 - s(t))x_2(t)\,,$$
$$\Pi_3(q_3(t), s(t), x_3(t), E(t)) = p_3 - kq_3(t)E(t)(1 - s(t))x_3(t)\,.$$

For simplicity denote

$$u_2(t) = q_2(t)E(t)\,,$$
$$u_3(t) = q_3(t)E(t)\,.$$

Hence the player's payoff becomes:

$$\begin{aligned} J &= g(x_2(T), x_3(T)) + \int_0^T e^{-\rho t}[-ku_2(t)^2(1-s(t))^2 x_2(t)^2 + p_2 u_2(t)(1-s(t))x_2(t) - c^0 u_2(t) - \\ &- ku_3(t)^2(1-s(t))^2 x_3(t)^2 + p_3 u_3(t)(1-s(t))x_3(t) - c^0 u_3(t)]dt\,, \end{aligned}$$

or

$$\begin{aligned} J = g(x_2(T), x_3(T)) &+ \int_0^T [-\tfrac{1}{2}au_2(t)^2(1-s(t))^2 x_2(t)^2 + b_2 u_2(t)(1-s(t))x_2(t) - cu_2(t) - \\ &- \tfrac{1}{2}au_3(t)^2(1-s(t))^2 x_3(t)^2 + b_3 u_3(t)(1-s(t))x_3(t) - cu_3(t)]dt\,, \end{aligned}$$

where

$$\begin{aligned} a &= 2k\exp(-\rho t)\,, \\ b_2 &= p_2 \exp(-\rho t)\,, \\ b_3 &= p_3 \exp(-\rho t)\,, \\ c &= c^0 \exp(-\rho t)\,. \end{aligned}$$

We are interested in the optimal solution of the following problem:

$$\begin{cases} \max(J(u_2(t), u_3(t)))\,, \\ \text{where } x_1(t), x_2(t), x_3(t) \text{ are defined in } (4)\,. \end{cases} \tag{5}$$

Theorem 2. *Let $u_2^*(t)$, $u_3^*(t)$, $x_1^*(t)$, $x_2^*(t)$, $x_3^*(t)$, $\lambda_2(t)$, $\lambda_3(t)$ satisfy the equation (4) and*

$$u_2^*(t) = \frac{(b_2 - \lambda_2(t))(1-s(t))x_2^*(t) - c}{a(1-s(t))^2 {x_2^*(t)}^2}\,,\ 0 \le t \le T\,,$$

$$u_3^*(t) = \frac{(b_3 - \lambda_3(t))(1-s(t))x_3^*(t) - c}{a(1-s(t))^2 {x_3^*(t)}^2}\,,\ 0 \le t \le T\,,$$

$$\lambda_1'(t) = \lambda_1(t)(\alpha_1 + \beta_1 + 2dx_1(t)) - \lambda_2(t)\alpha_1\,,\ 0 \le t \le T\,,$$

$$\begin{aligned} \lambda_2'(t) &= -u_2^*(t)(1-s(t))(b_2 - au_2^*(t)(1-s(t))x_2^*(t)) - \lambda_1(t)kl_2\sigma_2 + \\ &+ \lambda_2(t)(\beta_2 + \alpha_2 + u_2^*(t)(1-s(t))) - \lambda_3(t)\alpha_2\,,\ 0 \le t \le T\,, \end{aligned}$$

$$\begin{aligned} \lambda_3'(t) &= -u_3^*(t)(1-s(t))(b_3 - au_3^*(t)(1-s(t))x_3^*(t)) - \lambda_1(t)kl_3\sigma_3 + \\ &+ \lambda_3(t)(\beta_3 + u_3^*(t)(1-s(t)))\,,\ 0 \le t \le T\,, \end{aligned}$$

$$\lambda_i(T) = g'_{x_i}(x_1^*(T), x_2^*(T), x_3^*(T))\,, i = 1\,, 2\,, 3\,.$$

Then $u_2^*(t), u_3^*(t)$ is the solution of the problem (7).

From the system

$$\begin{cases} u_2(t) = q_2(t)E(t)\,, \\ u_3(t) = q_3(t)E(t)\,, \\ q_2(t) + q_3(t) = 0.02\,, \end{cases}$$

we can find

$$\begin{cases} E^*(t) = \frac{u_2^*(t) + u_3^*(t)}{0.02}\,, \\ q_2^*(t) = \frac{u_2^*(t)}{E^*(t)}\,, \\ q_3^*(t) = \frac{u_3^*(t)}{E^*(t)}\,. \end{cases}$$

So, if the center's strategy $s(t)$, $t \in [0, T]$ is known to the player, he can find the optimal behaviour $u_2^*(t), u_3^*(t)$. This, in turn, becomes known to center, which, depending on the player's strategy, receives either a gain or a loss.

We examine the following functionals determining center's gain:

1. $I_1 = -\int_0^T [(x_1(t) - \bar{x}_1(t))^2 + (x_2(t) - \bar{x}_2(t))^2 + (x_3(t) - \bar{x}_3(t))^2] dt$,

2. $I_2 = -\int_0^T (U(t) - \hat{x}(t))^2 dt$,

3. $I_3 = -\int_0^T |U(t) - \hat{x}(t)| \cdot \theta \cdot (\Pi_2(q(t), s(t), x_2(t), E(t)) + \Pi_3(q(t), s(t), x_3(t), E(t)))dt$,

where $U(t) = q_2(t)E(t)(1 - s(t))x_2(t) + q_3(t)E(t)(1 - s(t))x_3(t)$ – player's catch at time t, $\Pi_i(q_i(t), s(t), x_i(t), E(t)) = p_i - kq_i(t)E(t)(1 - s(t))x_i(t),\ p_i, k > 0$.

3.2. Game model for an artificially regenerated population

Let the population consist of three age classes: young fish, middle–age fish and old fish. The model is given for the artificially regenerated population.

The dynamics of the fishery is described by the system of equations [7]:

$$\begin{cases} x_1'(t) = k(l_2\sigma_2 x_2(t) + l_3\sigma_3 x_3(t)) - (\alpha_1 + \beta_1)x_1(t) - dx_1^2(t)\,, \\ x_2'(t) = \alpha_1 x_1(t) - (l_2 + \beta_2 + \alpha_2)x_2(t) - q_2(t)E(t)(1 - s(t))x_2(t)\,, \\ x_3'(t) = \alpha_2 x_2(t) - (l_3 + \beta_3)x_3(t) - q_3(t)E(t)(1 - s(t))x_3(t),\ 0 \le t \le T,\ x_i(0) = x_i^0\,, \end{cases} \tag{6}$$

where $x_1(t) \ge 0$ – number of young fish at time t, $x_2(t) \ge 0$ – number of middle–age fish at time t, $x_3(t) \ge 0$ – number of old fish at time t, l_2, l_3 – escapement rates, σ_2, σ_3 – mean fertility in the second and third age classes, k – egg survival rate, α_1, α_2 – recruitment rates, $\beta_1 + dx_1(t), \beta_2, \beta_3$ – mortalities, $E(t) \ge 0$ – firm's fishing efforts measured as the number of vessels involved in fishing at time t, $q_2(t)$, $q_3(t) > 0$, $q_2(t) + q_3(t) = 0.02$ – catchability coefficients for middle–age fish and old fish related to the unit fishing effort of the firm (fishing net's ratio for the two classes), $s(t)$ – proportionof the reserved area in the reservoir.

The player's revenue is

$$\begin{aligned} J \;&= g(x_2(T), x_3(T)) + \int_0^T e^{-\rho t}[\Pi_2(q_2(t), s(t), x_2(t), E(t))\cdot \\ &\cdot(q_2(t)E(t)(1 - s(t))x_2(t) + l_2 x_2(t)) + \Pi_3(q_3(t), s(t), x_3(t), E(t))\cdot \\ &\cdot(q_3(t)E(t)(1 - s(t))x_3(t) + l_3 x_3(t)) - c^0 E(t)(q_2(t) + q_3(t))]dt\,, \end{aligned}$$

where ρ – discount rate, c^0 – catching cost for one vessel and Π_i, $i = 2, 3$ – price functions for different age classes, specified as

$$\Pi_2(q_2(t), s(t), x_2(t), E(t)) = p_2 - kq_2(t)E(t)(1 - s(t))x_2(t) - l_2 x_2(t)\,,$$
$$\Pi_3(q_3(t), s(t), x_3(t), E(t)) = p_3 - kq_3(t)E(t)(1 - s(t))x_3(t) - l_3 x_3(t)\,.$$

For simplicity denote

$$u_2(t) = q_2(t)E(t)\,,$$
$$u_3(t) = q_3(t)E(t)\,.$$

Hence the player's payoff becomes:

$$\begin{aligned} J &= g(x_2(T), x_3(T)) + \int_0^T e^{-\rho t}[-ku_2(t)^2(1-s(t))^2 x_2(t)^2 + p_2 u_2(t)(1-s(t))x_2(t) - \\ &- l_2 u_2(t)(1-s)x_2^2(t) - kl_2 u_2(t)(1-s(t))x_2^2(t) + p_2 l_2 x_2(t) - l_2^2 x_2^2(t) - c^0 u_2(t) - \\ &- ku_3(t)^2(1-s(t))^2 x_3(t)^2 + p_3 u_3(t)(1-s(t))x_3(t) - l_3 u_3(t)(1-s)x_3^2(t) - \\ &- kl_3 u_3(t)(1-s(t))x_3^2(t) + p_3 l_3 x_3(t) - l_3^2 x_3^2(t) - c^0 u_3(t)]dt\,. \end{aligned}$$

Let's rewrite the payoff function in the following way

$$\begin{aligned} J &= g(x_2(T), x_3(T)) + \int_0^T [-\tfrac{1}{2}au_2(t)^2(1-s(t))^2 x_2(t)^2 + b_2 u_2(t)(1-s(t))x_2(t) - \\ &- \tfrac{1}{2}l_{22}u_2(t)(1-s)x_2^2(t) - \tfrac{1}{2}al_2 u_2(t)(1-s(t))x_2(t)^2 + b_2 l_2 x_2(t) - \\ &- \tfrac{1}{2}l_{22}l_2 x_2^2(t) - cu_2(t) - \tfrac{1}{2}au_3(t)^2(1-s(t))^2 x_3(t)^2 + \\ &+ b_3 u_3(t)(1-s(t))x_3(t) - \tfrac{1}{2}l_{33}u_3(t)(1-s)x_3^2(t) - \tfrac{1}{2}al_3 u_3(t)(1-s(t))x_3(t)^2 + \\ &+ b_3 l_3 x_3(t) - \tfrac{1}{2}l_{33}l_3 x_3^2(t) - cu_3(t)]dt\,, \end{aligned}$$

where

$$a = 2k\exp(-\rho t)\,,$$
$$b_2 = p_2\exp(-\rho t)\,,$$
$$b_3 = p_3\exp(-\rho t)\,,$$
$$l_{22} = 2l_2\exp(-\rho t)\,,$$
$$l_{33} = 2l_3\exp(-\rho t)\,,$$
$$c = c^0\exp(-\rho t)\,.$$

We are interested in the optimal solution of the following problem:

$$\begin{cases} \max(J(u_2(t), u_3(t)))\,, \\ \text{where } x_1(t), x_2(t), x_3(t) \text{ are defined in (6)}\,. \end{cases} \tag{7}$$

Theorem 3. *Let $u_2^*(t)$, $u_3^*(t)$, $x_1^*(t)$, $x_2^*(t)$, $x_3^*(t)$, $\lambda_2(t)$, $\lambda_3(t)$ satisfy the equation (6) and*

$$u_2^*(t) = \frac{(b_2 - \lambda_2(t) - \frac{1}{2}al_2 x_2 - \frac{1}{2}l_{22}x_2(t))(1-s(t))x_2^*(t) - c}{a(1-s(t))^2 {x_2^*(t)}^2}\,,\ 0 \le t \le T\,,$$

$$u_3^*(t) = \frac{(b_3 - \lambda_3(t) - \frac{1}{2}al_3 x_3 - \frac{1}{2}l_{33}x_3)(1-s(t))x_3^*(t) - c}{a(1-s(t))^2 {x_3^*(t)}^2}\,,\ 0 \le t \le T\,,$$

$$\lambda_1'(t) = \lambda_1(t)(\alpha_1 + \beta_1 + 2dx_1(t)) - \lambda_2(t)\alpha_1\,,\ 0 \le t \le T\,,$$

$$\begin{aligned} \lambda_2'(t) &= -u_2^*(t)(1-s(t))(b_2 - au_2^*(t)(1-s(t))x_2^*(t) - al_2 x_2^*(t) - l_{22}x_2^*(t)) - b_2 l_2 + \\ &+ l_{22}l_2 x_2^*(t) - \lambda_1(t)kl_2\sigma_2 + \lambda_2(t)(l_2 + \beta_2 + \alpha_2 + u_2^*(t)(1-s(t))) - \lambda_3(t)\alpha_2\,,\ 0 \le t \le T\,, \end{aligned}$$

$$\begin{aligned} \lambda_3'(t) &= -u_3^*(t)(1-s(t))(b_3 - au_3^*(t)(1-s(t))x_3^*(t) - al_3 x_3^*(t) - l_{33}x_3^*(t)) - \\ &- b_3 l_3 + l_{33}l_3 x_3^*(t)\lambda_1(t)kl_3\sigma_3 + \lambda_3(t)(l_3 + \beta_3 + u_3^*(t)(1-s(t)))\,,\ 0 \le t \le T\,, \end{aligned}$$

$$\lambda_i(T) = g'_{x_i}(x_1^*(T), x_2^*(T), x_3^*(T))\,, i = 1\,, 2\,, 3\,.$$

Then $u_2^*(t), u_3^*(t)$ is the solution of the problem (7).

From the system

$$\begin{cases} u_2(t) = q_2(t)E(t)\,, \\ u_3(t) = q_3(t)E(t)\,, \\ q_2(t) + q_3(t) = 0.02\,, \end{cases}$$

we can find

$$\begin{cases} E^*(t) = \frac{u_1^*(t)+u_2^*(t)}{0.02}\,, \\ q_2^*(t) = \frac{u_2^*(t)}{E^*(t)}\,, \\ q_3^*(t) = \frac{u_3^*(t)}{E^*(t)}\,. \end{cases}$$

We examine the following functionals determining the center's gain:

1. $I_1 = -\int_0^T [(x_1(t) - \bar{x}_1(t))^2 + (x_2(t) - \bar{x}_2(t))^2 + (x_3(t) - \bar{x}_3(t))^2]dt\,,$

2. $I_2 = -\int_0^T (U(t) - \hat{x}(t))^2 dt\,,$

3. $I_3 = -\int_0^T |U(t) - \hat{x}(t)| \cdot \theta \cdot (\Pi_2(q(t), s(t), x_2(t), E(t)) + \Pi_3(q(t), s(t), x_3(t), E(t)))dt\,,$

where $U(t) = q_2(t)E(t)(1-s(t))x_2(t)+l_2x_2(t)+q_3(t)E(t)(1-s(t))x_3(t)+l_3x_3(t)$ – player's catch at time t, $\Pi_i(q_i(t), s(t), x_i(t), E(t)) = p_i - kq_i(t)E(t)(1-s(t))x_i(t) - l_ix_i(t)\,,\ p_i, k > 0\,.$

In Section 4 the values of J, I_1, I_2 and I_3 are found for various $s(t)$ and Nash bargaining solution is obtained. Note that the results of numerical simulations for this case are almost the same as in the model analysed in the previous section.

4. Numerical modelling

4.1. Game model for one player and a constant $s(t)$

Modelling was carried out for the following values:

$$\begin{array}{lll} r = 0.06\,, & K = 300000\,, & \rho = 0.02\,, \\ k = 0.8\,, & p = 6000\,, & q = 0.002\,, \\ c^0 = 500000\,, & \alpha = 0.02\,, & T = 100\,. \end{array}$$

Let the initial size of the population be $x(0) = 150000$ and there is no reserved area at all $((s(t) = 0,\ t \in [0,T]))$. You can see the optimal values of $E^*(t)$ in Fig.1.1. The figure indicates that the number of vessels involved in fishing should be reduced from 8 to 6. The size of the population grows from 150000 to 230000 individuals (Fig.1.2), and the catch also increases from 2400 to 2900 individuals per time unit (Fig.1.3).

The player's profit, if he uses the optimal behavior is $J = 310903428.6$. If the size of the population which is optimal for reproduction is $\bar{x} = 180000$, then the population restoration cost for the centre is $I_1 = -153576318700$.

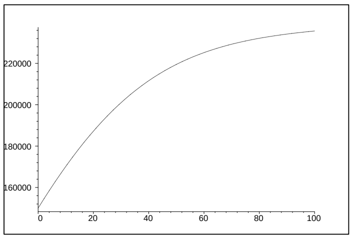

Figure 1.1. Values of $x^*(t)$

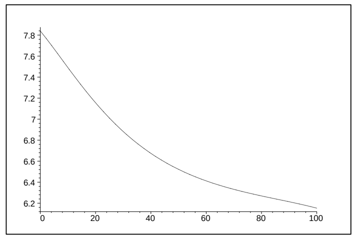

Figure 1.2. Values of $E^*(t)$

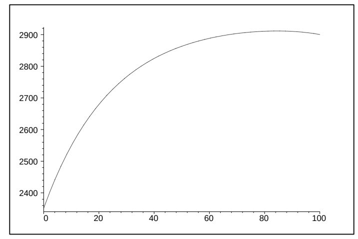

Figure 1.3. Values of $U^*(t)$

Now consider the case where 40 percents of the territory is reserved ($s(t) = 0.4$, $t \in [0, T]$). The optimal values of $E^*(t)$ are given in Fig.2.1. The number of vessels in the first 10 periods of time increases from 9, and then decreases to 8. The population growth is greater, than in the first scenario, from 150000 to 250000 individuals (Fig.2.2), and the catch increases from 1600 to 2500 individuals (Fig.2.3), which is less, than in the first case.

The player's payoff decreases $J = 221247734.8$, and for the same value $\bar{x} = 180000$, the population restoration cost for the centre also decreases $I_1 = -254620482600$.

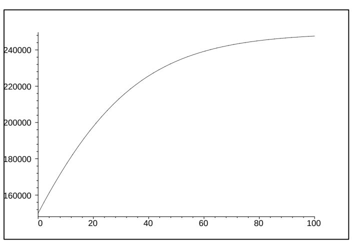

Figure 2.1. Values of $x^*(t)$

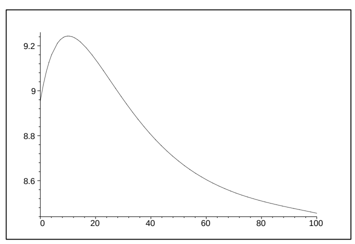

Figure 2.2. Values of $E^*(t)$

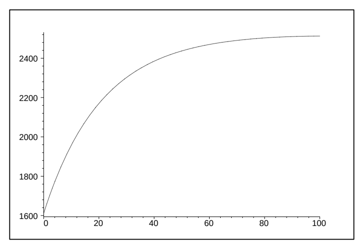

Figure 2.3. Values $U^*(t)$

For $s(t) = 0.8$, $t \in [0, T]$, the optimal values of $E^*(t), U^*(t), x^*(t)$ are shown in Fig.3.1–3.3. Interestingly, the opimal solution is no harvesting on the time interval $[0, 14]$, and after progressive raising of fishing efforts to 8 vessels. The catch grows from 0 to 900 individuals, and the size of the population increases from 150000 to 290000 individuals.

The player's profits for the same parameters are $J = 26999742$ and $I_1 = -745334490900$.

Thus, changing values of $s(t)$, the centre and the player get different profits. The Nash bargaining solution can be used to settle this conflict.

As the initial bargaining solution it is natural to choose the pair $(I^0, 0)$, where I^0 is center's cost, when no harvesting is done, and player's payoff is equal to zero.

In case 1) $I_1^0 = -\int\limits_0^T (x(t) - \bar{x})^2 dt$ is the cost for restoration of a naturally regenerating population, where the equation (1) is of the form $x'(t) = F(x(t))$, and in case 2) $I_2^0 = -\hat{x}^2\, T$ and 3) $I_3^0 = -\hat{x}\, p\, \alpha\, T$ are the costs for demand satisfaction.

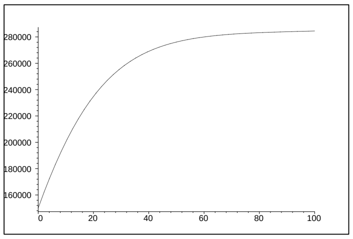

Figure 3.1. Values of $x^*(t)$

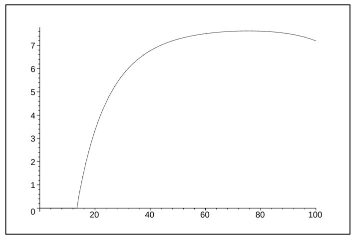

Figure 3.2. Values of $E^*(t)$

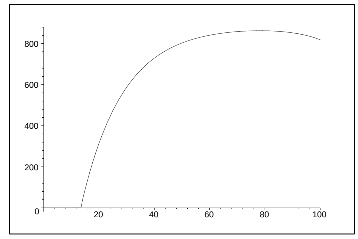

Figure 3.3. Values of $U^*(t)$

Values of $s(t)$, which correspond to optimal Nash solutions, are shown in Table 1.

Table 1. Optimal values of $s(t)$
Functional I_1

$\bar{x}(t)$	180000	200000	220000	240000	250000	260000	280000	300000	320000
Nash	0	0	0	0.3	0.55	0.7	0.8	0.9	0.9

For the functionals I_2, I_3, values of $s(t)$, which correspond to optimal Nash solutions, are shown in Table 2.

Table 2. Optimal values of $s(t)$
Functional I_2

$\hat{x}(t)$	500	700	1000	1200	1500	2000	2200	2500	3000
Nash	0.8	0.8	0.7	0.6	0.4	0	0	0	0

Functional I_3

$\hat{x}(t)$	500	700	1000	1200	1500	2000	2200	2500	3000
Nash	0.8	0.8	0.6	0.5	0.1	0	0	0	0

4.2. General case for one player

In the general case $s(t)$ can be an arbitrary function. We model the variant, in which the center can change the reserved portion of the reservoir one or more times at an arbitrary time moment.

Denote the moment, when the center can change it's policy as t^*, the reserved portion of the reservoir on the interval $[0, t^*)$ as $s_1(t)$, and on $[t^*, T]$ as $s_2(t)$.
You can see the results of simulations for uniformly distributed t^*, $s_1(t)$, $s_2(t)$ in Fig.4.

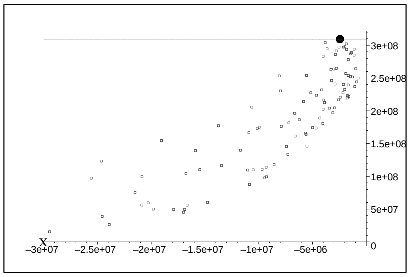

Figure 4. Values of I_3 and J

$x(0) = 150000 \ \hat{x} = 2500$

Nash equilibrium is achieved at $t^* = 95$

$s_1(t) = 0 \ \ s_2(t) = 0.7$

$I_3 = -2444747.669 \ J = 309403975$

4.3. Game model for an artificially regenerated population

Modelling was carried out for the following values:

$$
\begin{array}{lll}
T = 10\,, & k = 0.025\,, & l_2 = 0.012\ , \\
l_3 = 0.002\,, & \sigma_2 = 4800\,, & \sigma_3 = 5600\,, \\
\alpha_1 = 0.2\,, & \alpha_2 = 0.2\,, & \beta_1 = 0.36\,, \\
\beta_2 = 0.21\,, & \beta_3 = 0.54\,, & d = 0.0000000057\,, \\
\rho = 0.02\,, & k = 0.8\,, & p_2 = 6000\,, \\
p_3 = 12000\,, & c^0 = 500000/0.02\,, & \theta = 0.02\,.
\end{array}
$$

Let the initial sizes of the populations be $x_1(0) = 100000$, $x_2(0) = 50000$, $x_3(0) = 20000$ and there is no reserved area at all $((s(t) = 0,\ t \in [0, T]))$ You can see the optimal values of $E^*(t)$ in Fig.5.4. From the figure we observe that the number of vessels involved in fishing should be extend to 24. The number of young fish grows from 100000 to 175000 individuals (Fig.5.1), of middle–age fish – from 50000 to 70000 (Fig.5.2). The number of old fish decreases from 20000 to 11500 individuals in the first 4 time periods, and increases to 14000 individuals afterwards (Fig.5.3). The catch of middle–age fish increases to 2800 individuals per time unit (Fig.5.5). The catch of old fish decreases from 5800 to 4900 individuals in the first 4 time periods, and increases to 6000 individuals per time unit afterwards (Fig.5.6).

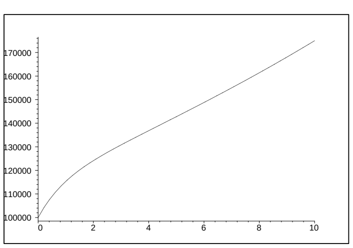

Figure 5.1. Values of $x_1^*(t)$

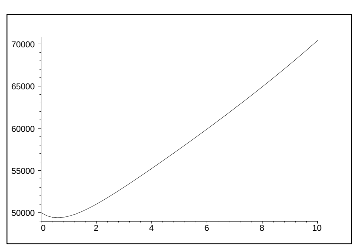

Figure 5.2. Values of $x_2^*(t)$

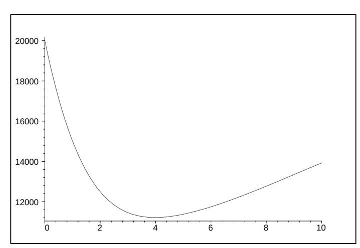

Figure 5.3. Values of $x_3^*(t)$

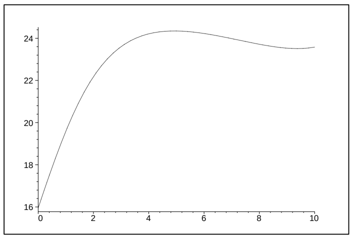

Figure 5.4. Values of $E^*(t)$

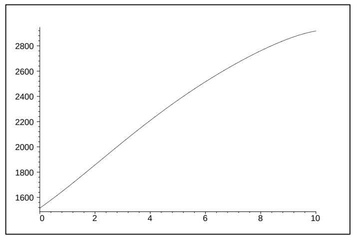

Figure 5.5. Values of $U_2^*(t)$

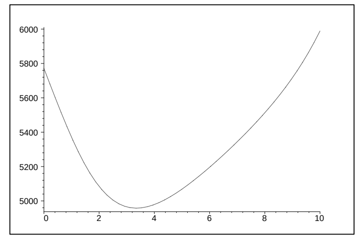

Figure 5.6. Values of $U_3^*(t)$

The player's payoff, if he uses the optimal behavior is $J = 431638465.7$. If the sizes of the age classes which are optimal for reproduction are $\bar{x}_1 = 150000, \bar{x}_2 = 100000$, $\bar{x}_3 = 30000$, then the population restoration cost for the centre is $I_1 = -28871013660$.

Now consider the case, when 50 percents of the territory is reserved ($s(t) = 0.5$, $t \in [0, T]$). The optimal values of $E^*(t)$ are given in Fig.6.4. The number of young fish grows more than in the first scenario: from 100000 to 180000 individuals (Fig.6.1), of middle–age fish – from 50000 to 75000 (Fig.6.2). The number of old fish decreases from 20000 to 13000 individuals in the first 4 time periods, and increases to 17000 individuals afterwards (Fig.6.3). The catch of middle–age fish increases to 2600 individuals (Fig.6.5). The catch of old fish decreases from 4700 to 3900 individuals in the first 4 time periods, and increases to 5200 individuals per time unit afterwards (Fig.6.6).

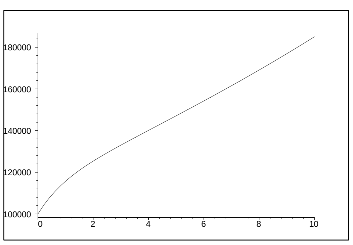

Figure 6.1. Values of $x_1^*(t)$

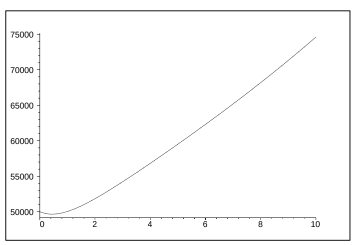

Figure 6.2. Values of $x_2^*(t)$

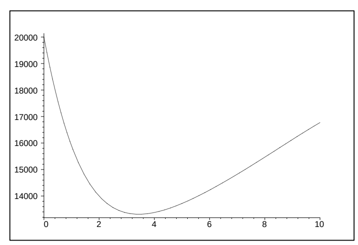

Figure 6.3. Values of $x_3^*(t)$

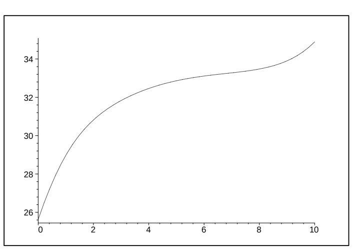

Figure 6.4. Values of $E^*(t)$

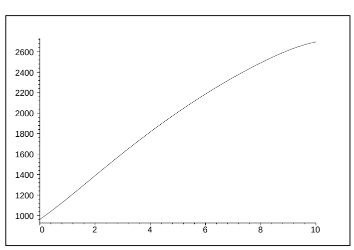

Figure 6.5. Values of $U_2^*(t)$

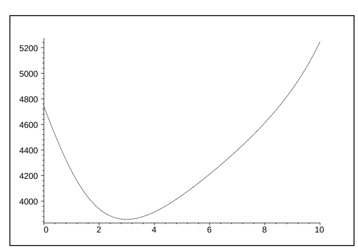

Figure 6.6. Values of $U_3^*(t)$

The player's payoff decreases to $J = 339883803.6$ and for the same values $\bar{x}_1 = 150000, \bar{x}_2 = 100000, \bar{x}_3 = 30000$, the population restoration cost for the centre decreases to $I_1 = -27598969290$.

For $s(t) = 0.9$, $t \in [0, T]$, the optimal values are shown in Fig.7.1–7.6. It is interesting to note, that the opimal solution is to catch no middle–age fish first 3 time periods, and an increase to 1200 individuals per unit of time afterwards (Fig.7.5) and to catch no old fish first 5 time periods, and an increase to 1400 individuals afterwards (Fig.7.6).

The player's profits for the same parameters are $J = 97221147.26$ and $I_1 = -30095273290$.

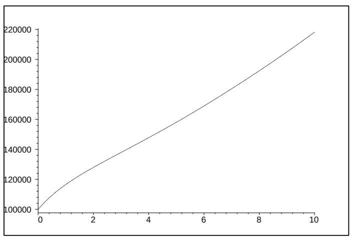

Figure 7.1. Values of $x_1^*(t)$

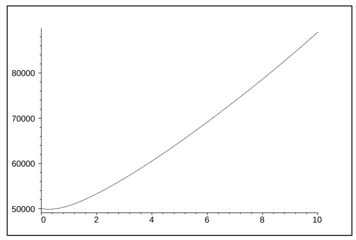

Figure 7.2. Values of $x_2^*(t)$

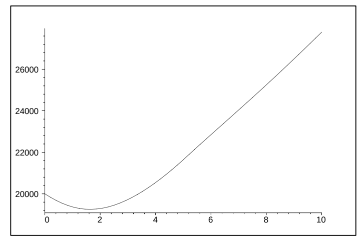

Figure 7.3. Values of $x_3^*(t)$

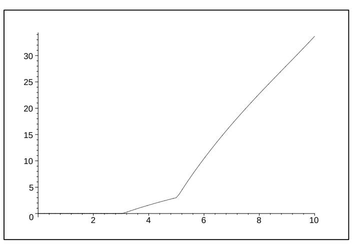

Figure 7.4. Values of $E^*(t)$

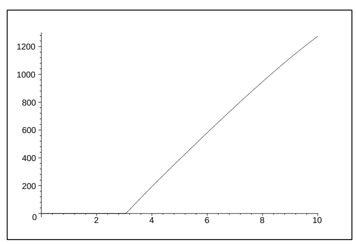

Figure 7.5. Values of $U_2^*(t)$

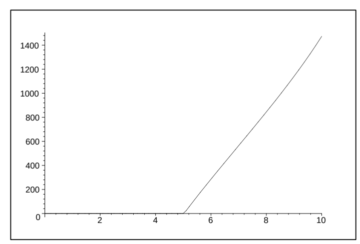

Figure 7.6. Values of $U_3^*(t)$

As the initial bargaining solution it is natural to choose the pair $(I^0, 0)$, where I^0 is center's cost, when no harvesting is done, and player's payoff is equal to zero.

In case 1) $I_1^0 = -\int\limits_0^T [(x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2 + (x_3(t) - \bar{x}_3)^2] dt$ is the cost for restoration of a naturally regenerating population, and in case 2) $I_2^0 = -\hat{x}^2 T$ and 3) $I_3^0 = -\hat{x}\,(p_2 + p_3)\,\theta\, T$ are the costs of satisfying the demand.

Values of $s(t)$, which correspond to optimal Nash solutions, are shown in Table 3–4.

Table 3. Optimal values of $s(t)$

Functional I_1

$\bar{x}_1(t)$	100000	150000	150000	150000	150000	200000	200000	200000	200000
$\bar{x}_2(t)$	50000	60000	70000	100000	120000	70000	70000	70000	100000
$\bar{x}_3(t)$	20000	15000	20000	30000	20000	20000	25000	30000	30000
Nash	0	0	0	0.5	0.6	0.9	0.9	0.9	0.9

Table 4. Optimal values of $s(t)$

Functional I_2

$\hat{x}(t)$	2000	3000	4000	4500	5000	5500	5700	6000	7000
Nash	0.9	0.8	0.6	0.5	0.4	0.1	0	0	0

Functional I_3

$\hat{x}(t)$	2000	3000	4000	4500	5000	5500	5700	6000	7000
Nash	0.8	0.8	0.7	0.6	0.4	0.1	0	0	0

4.4. General case with three age classes

The results of simulations for the case with three age classes are presented in Fig.8–10.

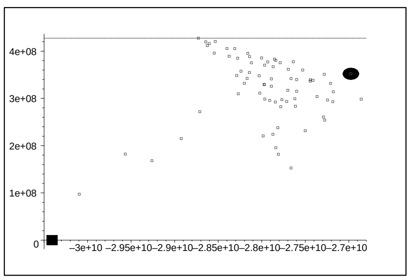

Figure 8: Values of I_1 and J

$$x_1(0) = 100000 \; \bar{x}_1 = 150000$$
$$x_2(0) = 50000 \; \bar{x}_2 = 100000$$
$$x_3(0) = 20000 \; \bar{x}_3 = 30000$$

Nash equilibrium is achieved at $t^* = 3$

$$s_1(t) = 0.3 \quad s_2(t) = 0$$
$$I_1 = -26976400230 \; J = 352132946.8$$

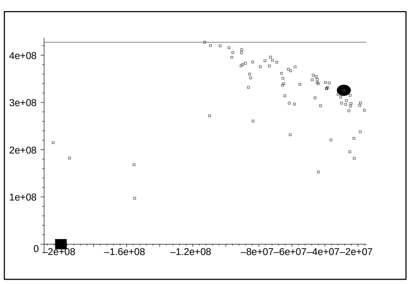

Figure 9: Values of I_2 and J

$$x_1(0) = 100000 \; x_2(0) = 50000$$
$$x_3(0) = 20000 \; \hat{x} = 4500$$

Nash equilibrium is achieved at $t^* = 3$

$$s_1(t) = 0.1 \quad s_2(t) = 0.7$$
$$I_2 = -28572831.16 \; J = 325762113.4$$

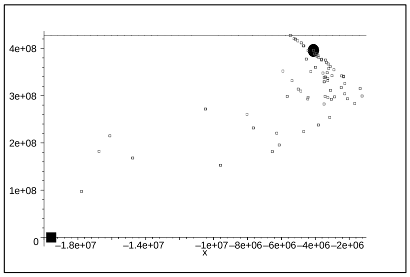

Figure 10: Values of I_3 and J

$$x_1(0) = 100000 \; x_2(0) = 50000$$
$$x_3(0) = 20000 \; \hat{x} = 5500$$

Nash equilibrium is achieved at $t^* = 4$

$$s_1(t) = 0.6 \;\; s_2(t) = 0.1$$
$$I_3 = -4111816.166 \; J = 395847353.3$$

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REFERENCES

1. C.W. Clark, *Bioeconomic modelling and fisheries management*, Wiley, NY, 1985.

2. H. Ehtamo, R.P. Hamalainen, *A cooperative incentive equilibrium for a resource management problem*, J. of Economic Dynamics and Control **17** (1993), 659–678.

3. R.P. Hamalainen, V. Kaitala, A. Haurie, *Bargaining on whales: A differential game model with Pareto optimal equilibria*, Oper. Res. Letters **3**, no.**1** (1984), 5–11.

4. A. Haurie, B. Tolwinski, *Acceptable equilibria in dynamic games*, Large Scale Systems **6** (1984), 73–89.

5. A.A. Petrosyan, V.V. Zakharov, *Mathematical models in ecology*, SPSY, 1997, (in russian).

6. L.S. Pontryagin, V.G. Boltyanski, R.V. Gamrelidze, E.F. Michenko, *Mathematical theory of optimal processes* , Nauka, 1969, (in russian).

7. V.I. Gurman, G.N. Konstantinov, *Planning and prediction of natural– economic systems* , Nauka, Novosibirsk, 1984, (in russian).

8. V.V. Mazalov, A.N. Rettieva, *About one resource management problem* , Applied and industrial mathematics review **9**, no.**2** (2002), TVP, Moscow, 293–306, (in russian).