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## EXTENDED ABSTRACTS

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## Foreword

The present volume contains full papers and extended abstracts accepted for the Workshop "Networking Games and Management" held in the Institution of the Russian Academy of Sciences, Institute of Applied Mathematical Research KarRC RAS, Petrozavodsk, Russia, June 28-30, 2009.

The emphasis of the seminar is on the following topics:

- networking games and management,
- optimal routing,
- price of anarchy,
- auctions,
- negotiations,
- learning and adaptive games, etc.

16 papers from Finland, Japan, Poland and Russia were submitted and included into the volume.

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On behalf of the Organization Comittee
Professor Vladimir V. Mazalov

# Contents

# A game theoretical model of tax auditing with using a statistical information about taxpayers

V. M. Bure, S. Sh. Kumacheva

St.-Petersburg State University, Saint-Petersburg, Russia

At the heart of this model there is the hierarchical game ([5]), in which tax authority and finite number of taxpayers are players. Due to the tradition, founded by [2], [3], [6] we will consider the interaction of the tax authority to each taxpayer due to the scheme principal-to-agent.

There are $n$ taxpayers, each of them has an income $i_k$, where $k = \overline{1,n}$. Taxpayer $k$ can declare his incomes level $r_k$ and $r_k \leq i_k$ for every $k = \overline{1,n}$. Let $t$ be tax rate $\pi$ – penalty rate; they are measured as the parts of some amount of money. Tax auditing of the taxpayer $k$ is made by the tax authority with probability $p_k$, where $k = \overline{1,n}$. The model is built in an assumption that these probabilities are known by taxpayers. Audit is supposed to reveal evasions always.

As a result of a tax audit, that revealed a tax evasion, the taxpayer must pay the underpaid tax and the penalty; both of them depend on the evasions level. Four kinds of penalties are known from papers [2], [3]. In the simplest case, when the penalty is proportional to evasion, the taxpayer $k$ must pay: $(t + \pi)(i_k - r_k)$. The expected tax payment of the taxpayer $k$ is:

$$u_k = tr_k + p_k(t + \pi)(i_k - r_k),$$

where the first summand is always paid by the taxpayer (pre-audit payment), and the second - as the result of the tax auditing, made with probability $p_k$ (post-audit payment). The expected payoff $b_k$ of the taxpayer $k$ is:

$$b_k = i_k - u_k = i_k - tr_k - p_k(t+\pi)(i_k - r_k).$$

Every taxpayers aim is to maximize his payoff function $b_k$.

Let $c$ be the cost of one audit. The tax authoritys net income consists of taxation (taxpayers payments corresponding to their declared income), taxes on the evasion level and penalties (as the audit results) less total audit cost. Being the sum of tax payments got from every taxpayer, the expected tax authoritys net income can be calculated as the difference between expected tax payments of $n$ taxpayers and expected cost of audit of $n$ taxpayers:

$$R = \sum_{k=1}^{n} R_k = \sum_{k=1}^{n}(u_k - c_k) = \sum_{k=1}^{n}(tr_k + p_k(t+\pi)(i_k - r_k) + p_k c).$$

The tax authoritys aim is to maximize its expected income $R$.

A taxpayers strategy is to make a decision to evade or not to evade, i.e. to declare $r_k < i_k$ or $r_k = i_k$. A tax authoritys strategy is to choose the optimal (in order to maximize the income) combination of the quantities $(t, pi, p_k)$ – some optimal contract ([2], [3]).

This game is considered in assumption the players are risk neutral. Therefore, making decision to evade or not (choosing the strategy), the taxpayer compares the quantities $ti_k$ (profit less taxes as a result of declaring of the true income) and $u_k$ (the expected loss as a result of auditing) and then models the best answer on the tax authoritys expected actions in every situation.

Much more difficult is to estimate the tax authoritys expected income: it doesnt know the exact meaning of every taxpayers true income. Taking in consideration this circumstance, several mathematical models ([1], [2], [3], [6]) consider as an additional factor in choosing a strategy of the tax authority the disposed information about taxpayers income distribution or statistical information about each taxpayers income as a result of monitoring. This model assumes the probability of taxpayers tax evasion is beta-distributed. In analogy of a credit story a taxpayers ¡¡tax story¿¿ is considered – Bernoulli-distributed replicate sample, which characterize a taxpayers behavior in the previous tax periods (tax authority has such information as a rule). Then, using the feature of conjugate families of distribution, the tax authority can conclude, that the taxpayer $k$ audit is necessary and choose the appropriate probability

Taking into consideration the last feature we built and analyzed the graphics of players payoff functions, depending on audit probabilities $b_k(p_k)$ and $R_k(p_k)$. The optimal (in order to maximize the payoff functions) strategies and equilibrium are found.

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# Equilibria in Bayesian Splittable Traffic Routing Game

Julia V. Chuyko

Institute of Applied Mathematical Research,
Karelian Research Centre RAS, Petrozavodsk, Russia

E-mail: julia@krc.karelia.ru

Consider Bayesian routing game $\Gamma = \langle n, m, f, T, p\rangle$ in network with $n$ selfish users and $m$ parallel links, where each user chooses his route trying to minimize the expected delay of his own traffic he send. Delays are based on player-specific capacities $f_{ie}(x) = a_{ie}x$. Each user has a set of traffic types $T_i$ and a joined distribution $p(t_1, \ldots, t_n)$ of users' traffic types is known. Traffic amounts $w_i(t)$ we suppose to be encoded in $T$. In the model each user $i$ knows only his traffic type $t_i$ that he is going to send and joined type distribution, so he can find conditional distribution depending on his traffic type $p(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n | t_i = t) = \frac{p(t_1,\ldots,t,\ldots,t_n)}{p(i,t)}$, where $p(i,t) = \sum\limits_{(t_1,\ldots,t_n)\in T: t_i = t} p(t_1, \ldots, t_n)$ is a probability that user $i$ sends traffic of type $t$.

Strategies profiles in the game are $x = \{x_i^{te}\}_{i\in[n], t\in T_i, e\in[m]}$ where $x_i^{te}$ is $i$-th user's traffic of type $t$ to send it throw link $e$. They must be non-negative and $\sum\limits_{e\in[m]} x_i^{te} = w_i(t)$.

An expected load of link $e$ we can find as $\delta_e(x,p) = \sum\limits_{(t_1,\ldots,t_n)\in T} p(t_1, \ldots, t_n) \sum\limits_{i\in[n]} x_i^{t_i e}$. Since user doesn't know traffic types that others send, he need to use conditional expected loads to define his costs depending his own behaviour. Conditional expected load of link $e$ is $\delta_e(x, (p|t_k = t)) = \delta_e^{-k}(x, (p|t_k = t)) + x_k^{te}$, where $\delta_e^{-k}(x, (p|t_k = t)) = \sum\limits_{(t_1,\ldots,t_n)\in T: t_k = t} p(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n | t_k = t) \sum\limits_{i\in[n]\setminus\{k\}} x_i^{t_i e}$ is conditional expected load from other users than $k$.

So, conditional expected costs for user $k$ sending traffic of type $t$ are $v_{(k,t)}(x,p) = \max_{e \in [m]: x_k^{te} > 0} f_{ke}(\delta_e(x,(p|t_k = t)))$ and his expected costs are $PC_k(x,p) = \sum_{t \in T_k} p(k,t) v_{(k,t)}(x,p)$. Note that each component of the sum doesn't depend on other traffic types of the user $k$.

The objects of the research are equlibria: Wardrop Equilibrium, that always exists and can be found using potential function, and its special case Bayesian Wardrop Equilibrium, that can be more easily understood by users, but its existence is an open question.

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# Сетевые игры и игры на сетях

М. В. Губко, Д. А. Новиков, А. Г. Чхартишвили

Учреждение Российской академии наук
Институт проблем управления
им. В.А. Трапезникова РАН, Москва

**Игры и графы.** Между такими развитыми разделами прикладной математики, как теория игр и теория графов, существует глубокая взаимосвязь. Можно привести множество примеров использования конструкций и результатов теории графов в игровых постановках: дерево задает структуру принятия решений в игре в развернутой форме [11]; граф (вершины - игроки) задает структуру возможных коалиций [14]; на графе в дискретном времени осуществляется «игра поиска» (вершины – позиции игроков, ребра – возможные пути переходов) [10]; ориентированный граф описывает, от чьих действий зависят выигрыши агентов (для реализуемости равновесия Нэша достаточно связности графа), в более общем случае граф отражает структуру информированности игроков [9] или структуру коммуникаций между ними [8]; граф отражает постоянные или временные связи (информационные, технологические, подчиненности и т.п.) между игроками [7]...

**Сетевые игры.** Отдельно следует упомянуть теорию сетевых игр – относительно молодой (развивающийся с конца 70-х годов XX века) раздел теории игр, акцентирующий внимание как раз на формировании структур – устойчивых связей между игроками – в условиях несовпадения интересов и/или различной информированности последних (см. обзоры [3] и [13]).

Концепции решения сетевых игр удачно сочетали в себе элементы кооперативного и некооперативного подходов – специфика задачи позволяла рассматривать только парные взаимодействия («коалиции» из двух игроков) [13]. В то же время, применение таких игр в экономических задачах показало, что перечисления связей зачастую недостаточно для описания ситуации – каждая связь «отягощена» набором числовых параметров (например, объемами и ценами передаваемых товаров в играх формирования торговых сетей [4]), также являющихся следствием выбора игроков. Моделью сети при этом становится взвешенный граф, а спецификой сетевых игр, отличающей их от «игр вообще» остается то, что выборы и выигрыши игроков описываются характеристиками попарных взаимодействий (связей) между игроками.

**Игры на сетях.** В последние годы все чаще появляются содержательные постановки задач описания и исследования такого взаимодействия игроков, что результат их взаимодействия (или связь между выбираемыми действиями или стратегиями и выигрышами) определяется той или иной «сетевой» (теоретико-графовой) моделью. Такого рода игры будем называть *играми на сетях*. Приведем два примера.

*«Когнитивные игры»* [6], в которых когнитивная карта [12] – взвешенный ориентированный граф (его вершинами являются факторы, значения которых измеряются в непрерывной или нечеткой шкале, а взвешенными или функциональными дугами описывается взаимовлияние факторов) – используется для учета причинно-следственных связей и взаимовлияния факторов, а также для моделирования динамики слабоформализуемых систем. Например, описав взаимосвязь между факторами в виде системы линейных дифференциальных уравнений второго порядка и задав начальные значения, можно анализировать динамику факторов, «установившиеся» значения и т.д., рассматривая все эти аспекты с точки зрения лиц, заинтересованных в том или ином развитии ситуации, или исследуя несовпадение целей различных субъектов. Имея модель связи между факторами можно рассматривать игровую постановку – пусть игроки имеют возможность влиять на начальные значения факторов (например, для каждого игрока задано множество «контролируемых» им факторов), а их выигрыши зависят от «установившихся» значений факторов. Пример линейной игры такого рода рассмотрен в [6].

*«Игры на социальных сетях»* [2], в которых вершинами являются агенты – участники социальной сети, а взвешенные дуги отражают степени их «доверия» друг другу. Мнение каждого агента формируется под влиянием его начального мнения и мнений других агентов с учетом их доверия друг другу (динамика мнений описывается системой линейных дифференциальных уравнений первого порядка). По-

мимо агентов, в модели существуют игроки, которые могут влиять на агентов и их взаимодействие. Зная связь между начальными мнениями, а также структурой социальной сети, и итоговыми мнениями, можно ставить и решать задачу формирования игроками таких начальных мнений у агентов и таких связей между ними (включая как структуру, так и степени доверия), которые были бы равновесием (в том или ином смысле) соответствующей игры.

Общим для приведенных двух примеров, да и, пожалуй и для других игр на сетях, является следующее. Связь между действиями игроков и результатом, который определяет их выигрыши, описывается в рамках достаточно простой сети динамической системой. Дальше все сводится к анализу свойств динамической системы, а затем – к той или иной классической теоретико-игровой постановке (в общем случае – к динамической игре [6]).

**Взаимосвязь между играми на сетях и сетевыми играми**[1]. Различие между сетевыми играми и играми на сетях состоит в том, что в первых предметом выбора игроков являются переменные, относящиеся к парному взаимодействию между игроками, а в играх на сетях – переменные, описывающие вершины сети (значения факторов в играх на когнитивных картах, мнения агентов – в играх на социальных сетях...). Однако считая «вершинные» переменные относящимися к петлям взвешенного графа, эти модели можно формально объединить. Польза же от такого объединения велика, поскольку во многих играх формирования сетей (например, в моделях информационных коммуникаций в многоагентных системах [1]) для расчета выигрышей игроков требуется привлекать модель сетевой динамики, как и в играх на сетях. Объединение моделей приведет к двухэтапной игре, на первом этапе которой игроки формируют сеть, а на втором этапе используют сформированную сеть для передачи информации, ресурсов и т.д. в соответствии с концепцией игр на сетях.

В докладе также рассматриваются этот и другие возможные способы объединения двух описанных классов игр, приводится общая

[1] Термин «сетевые игры» (network games) все чаще замещается термином «игры формирования сетей» (network formation game), более соответствующим сути игры, результатом которой является сеть, связывающая игроков. Эта тенденция имеет свое обоснование – сетевые игры могут рассматриваться как включающие в себя игры формирования сетей и игры на сетях, причем в последних сеть фиксирована.

постановка задачи управления формированием сетей, формулируется ряд содержательных моделей и описываются результаты их исследования.

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# Теоретико-игровые задачи управления в линейных социальных сетях

Д.А. Губанов, А.Г. Чхартишвили

Учреждение Российской академии наук
Институт проблем управления
им. В.А. Трапезникова РАН, Москва

**Социальные сети.** Социальная сеть – это граф, вершинами которого являются индивидуумы, а ребрами – социальные отношения между ними. Социальные сети существовали всегда в человеческом обществе, их модели используются, например, в различных направлениях экономического анализа (см. [1]). С развитием Интернета и так называемых онлайновых социальных сетей (таких как Livejournal – www.livejournal.com, Хабрахабр – habrahabr.ru и пр.) их роль, по-видимому, будет возрастать.

Из социальной психологии (см., напр., [2]) известно, что мнение индивидуума в социальной сети в значительной мере определяется мнением влиятельных для него соседей. Зная это, некто за пределами сети или внутри нее для достижения своих целей может попытаться изменить мнения небольшого множества ключевых пользователей в популярных онлайновых социальных сетях, посредством которых произойдет распространение мнений по всей сети. Предметом данного доклада является формирование мнений в социальной сети, моделируемое при помощи марковских цепей (о марковских цепях см., напр., гл. VIII [3]).

**Непосредственное и косвенное влияние.** Пусть элементы из множества $N = 1, \ldots, n$ – агенты – образуют социальную сеть. Обозначим неотрицательным числом $t_{ij}$ степень доверия $i$-го агента $j$-му (степень влияния $j$-го агента на $i$-го). Здесь и далее мы будем говорить как о влиянии, так и о доверии. Будем считать, что эти два понятия соотносятся следующим образом: выражение «степень доверия $i$-го агента $j$-му равна $t_{ij}$» тождественно по смыслу выражению «степень влияния $j$-го агента на $i$-го равна $t_{ij}$». Матрицу $t = (t_{ij})$ будем называть матрицей непосредственного доверия (влияния).

Доверие в социальной сети можно наглядно изображать в виде стрелок с весами, соединяющих вершины (например, стрелка от $i$-го агента к $j$-му с весом $t_{ij}$ означает, что соответствующую степень доверия). Будем считать выполненным условие нормировки: сумма весов всех исходящих стрелок каждого агента равна 1 (тем самым, матрица $t$ является стохастической).

Если $i$-й агент доверяет $j$-му, а $j$-й доверяет $k$-му, то это означает следующее: $k$-й агент косвенно влияет на $i$-го (хотя $i$-й может даже не знать о его существовании). Это соображение побуждает к поиску ответа на вопрос о том, кто в итоге формирует мнение в социальной сети.

**Структура результирующих влияний.** Пусть у каждого агента в некий начальный момент времени имеется мнение по некоторому вопросу, мнение $i$-го агента отражает вещественное число $b_i$. Мнение всех агентов сети отражает вектор-столбец мнений $b$ размерности $n$. Агенты в социальной сети взаимодействуют, обмениваясь мнениями. Этот обмен приводит к тому, что мнение каждого агента меняется в соответствии с мнениями агентов, которым данный агент доверяет. Будем считать это изменение линейным: мнение агента в следующий момент времени является взвешенной суммой мнений агентов, которым он доверяет (весами являются степени доверия $t_{ij}$). Отметим, что агент может доверять, в том числе, и самому себе: $t_{ii} > 0$.

Нетрудно убедиться, что в векторной записи измененное мнение агентов становится равным произведению матрицы непосредственного доверия на вектор начальных мнений: $t_b$. Если обмен мнениями продолжается и далее, то вектор мнений агентом становится равным $t^2b, t^3b$ и т.д.

Ели взаимодействие агентов продолжается достаточно долго, то их мнения стабилизируются – сходятся к результирующему мнению

$B = Tb$,

где $b$ – вектор начальных мнений, $T$ – матрица результирующего влияния, $t^n \to T$ при $n \to \infty$ (об условиях существования предела см. ниже), $B$ – вектор итоговых мнений.

Множество агентов, прямо или косвенно влияющих друг на друга, будем называть группой.

*Определение.* Группа – множество агентов, каждый из которых влияет (прямо иди косвенно) на любого агента из этого множества.

Справедливы следующие утверждения.

*Утверждение 1.* Каждый агент либо входит ровно в одну группу, либо не входит ни в одну.

*Утверждение 2 (достаточное условие стабилизации мнений).* Если в каждой группе существует хотя бы один агент $i$, для которого $t_{ii} > 0$, то мнения стабилизируются.

Таким образом, наряду с агентами, входящими в ту или иную группу, в сети существуют спутники – агенты, не входящий ни в одну группу.

Оказывается, что структура результирующих влияний (в случае стабилизации мнений) устроена следующим образом:

1) в каждой группе итоговые мнения элементов совпадают, т.е. каждая группа имеет общее мнение (которое можно считать мнением группы);

2) итоговые мнения спутников определяются только мнениями групп, т.е. начальные мнения спутников не оказывают никакого влияния на результирующие мнения каких-либо агентов.

**Управление и игра в социальной сети.** Помимо агентов, участниками модели могут являться другие субъекты, тем или иным образом заинтересованные в оказании влияния на результирующие мнения агентов – будем называть их игроками. Если игрок один (в этом случае его можно рассматривать как некий управляющий орган, воздействующий на сеть), то возможна постановка задачи управления: путем воздействия на начальные мнения агентов добиться выгодных центру результирующих мнений. Если игроков несколько, и каждый из них может воздействовать на агентов, то ситуация допускает формализацию и исследование в терминах теории игр: каждый из игроков стремиться максимизировать свою целевую функцию, определенную на множестве результирующих мнений агентов.

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# Index of riskiness for finit game

Anton S. Gurevich

The purpouse of this work is spreading of "measure of riskinnes", proposed by Foster & Hart for gambles, to the random variable and finite games.

## 1 Foster & Hart measure of riskiness

Everything contained in this section was presented in articles [1] and [2].

**Definition 1** *Gamble is the random variable $g \in R$, satisfying the following conditions, $E(g) > 0$, $P(g < 0) > 0$.*

**Definition 2** $L(g) = -min(g)$ *is the maximal loss of* $g$. $L(g) > 0$.

**Definition 3** *Real number $R(g) > 0$, uniquely determined by the equation $E\left[\log(1 + \frac{g}{R(g)})\right] = 0$, on $[L(g); \infty)$ is measure of riskiness for gamble $g$.*

Now suppouse $g$ and $h$ are gambles.

**Axiom 1** *If $g$ and $h$ have the same distribution then $Q(g) = Q(h)$.*

**Axiom 2** *If $g \leq h$ and $g \neq h$ then $Q(g) > Q(h)$.*

**Axiom 3** $Q(\lambda g) = \lambda Q(g)$ *for every* $\lambda > 0$.

**Axiom 4** $Q(g) + g \geq 0$.

**Axiom 5** *If for every value $x$ of $g$ either $h_{\|g=x} \equiv 0$ or $h_{\|g=x}$ is a gamble with $Q(h_{\|g=x}) = Q(g) + x$, then $Q(g + h) = Q(g)$.*

**Theorem 1** *The minimal function that satisfies axioms 1–5 is the measure of riskiness $R$.*

## 2 Spreaded index of riskines

To have the opportunity to work with random variable, we have to define spreaded index of riskines.

**Definition 4**

$$R^+(g) = \begin{cases} +\infty & \text{for } E(g) \le 0, P[g < 0] > 0, \\ R(g) & \text{for } E(g) > 0, P[g < 0] > 0, \\ 0 & \text{for } P[g < 0] = 0. \end{cases}$$

*is the spreaded index of riskiness*

**Axiom 6** *If $g \le h$ and $g \ne h$, then $Q(g) \ge Q(h)$.*

**Axiom 7** $P[Q(g) + g \ge 0] = 1$.

**Theorem 2** *Spreaded index of rickiness $R^+$ satisfies axioms 1,3,5–7.*

## 3 Index of riskines for finite game

Now introduce a model of games of many players. Let us have a set of players $I = \{1, \dots, n\}$, also have infinite sequence of finite games $G_1, G_2, G_3, \dots$. In every moment $t \in N$ players are offered a finite game $G_t = (I, Y^t, K^t)$, $Y = \{Y_1, \dots, Y_n\}$ – set of strategies for each player, $K = \{K_1, \dots, K_n\}$ – gain functions of players. Let every player $i$ offers game $G_t^{\lambda_i} = (I, Y^t, \lambda_i K^t)$, $\lambda_i \ge 0$. Let $\lambda = \min_{i \in N}(\lambda_i)$ will be selected and $G_t^\lambda = (I, Y, \lambda K)$ will be played. Though the Nash equilibrium situation $Z(G_t^\lambda)$ will be realized, and player $i$ gain $F_i(\lambda, t) = \lambda K_i(Z(G_t^\lambda))$. Players also have wealth $W_i(t)$, $W_i(1) > 0$, dynamics: $W_i(t+1) = W_i(t) + F_i(\lambda, t)$. $W(t) = (W_1(t), W_2(t), \dots, W_n(t),)$

Let the goal of players is avoiding bankruptcy: $W_i(t) > 0$ for any $t \in N$, and

$$P\left[\lim_{t \to \infty} W_i(t) = 0\right] = 0.$$

**Definition 5** *Global strategy $S$ is unambiguous mapping from set of all pairs $(W, G)$ to* $[0, +\infty)$

**Definition 6** *Strategy $S^*$ guarantees no-bankruptcy if it yields avoiding bankruptcy with probability 1 for every process $G$ and every initial wealth $W > 0$*

**Definition 7** $R_i^{++}(G) = R^+(F_i(1))$ *is index of riskiness for finit game $G$ for player $i$.*

**Theorem 3** *For any finite game $G_t$, any wealth $W_i(t)$ and any player $i$ exist unique number $\Lambda_i(G_t, W_i(t))$ such that strategy $S^*$ guarantees no-bankruptcy if and only if $S^*(W_i(t), G_t) \leq \Lambda_i(G_t, W_i(t))$. Where*

$$\Lambda_i(G_t, W_i(t)) = \begin{cases} +\infty & \textit{for } P[F_i^t(1) < 0] = 0, \\ \frac{W_i(t)}{R^+(F_i^t(1))} & \textit{for } E(F_i^t(1)) > 0, P[F_i^t(1) < 0] > 0, \\ 0 & \textit{for } E(F_i^t(1)) \leq 0, P[F_i^t(1) < 0] > 0. \end{cases}$$

Let $W(t) = W^* = (1, \dots, 1)$. Then

$$\Lambda_i^*(G) = \Lambda_i(G, W_i^*(t)) = \begin{cases} +\infty & \text{for } P[F_i(1) < 0] = 0, \\ \frac{1}{R^+(F_i(1))} & \text{for } E(F_i(1)) > 0, P[F_i(1) < 0] > 0, \\ 0 & \text{for } E(F_i(1)) \leq 0, P[F_i(1) < 0] > 0. \end{cases}$$

$$\Lambda^*(G) = \min_{i \in N}(\Lambda_i^*(G, W_i^*))$$

**Definition 8** *Player $j$ such that $\Lambda_j^*(G) = \Lambda^*(G)$, is minimizes player in finite game $G$.*

**Definition 9** $R^{++}(G) = R^+(F_j(1))$, *where $j$ – minimizes player is index of riskiness for finit game $G$.*

Index of riskiness for finit game has the following properties.

**Proposition 1** *If $W(t) = kW^*$, then*

$$\Lambda_i^*(G) = \Lambda_i(G, W_i^*(t)) = \begin{cases} +\infty & \textit{for } P[F_i(1) < 0] = 0, \\ \frac{k}{R^+(F_i(1))} & \textit{for } E(F_i(1)) > 0, P[F_i(1) < 0] > 0, \\ 0 & \textit{for } E(F_i(1)) \leq 0, P[F_i(1) < 0] > 0. \end{cases}$$

**Proposition 2** *For every finite game* $R^{++}(G^{\lambda}) = \lambda R^{++}(G)$.

**Proposition 3** *If* $G^{'} = (N, Y, K^{'})$, $G^{''} = (N, Y, K^{''})$, *where* $K^{'} = K^{''} + C$, $C \geq 0$, *then* $R^{++}(G^{'}) \leq R^{++}(G^{''})$.

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# Random priority zero-sum best choice game with disorder

Evgeny E. Ivashko

Institute of Applied Mathematical Research,
Karelian Research Centre RAS, Petrozavodsk, Russia

E-mail: ivashko@krc.karelia.ru

**Abstract**

A zero-sum version of the best-choice game with disorder is considered. Two players observe sequentially iid random variables with a known continuous distribution. In random time the distribution of observation is changed. The random variables cannot be perfectly observed. Players may use different values of levels in every step. After each sampling players take a decision for acceptance or rejection of the observation. If both want to accept the same observation then a random assignment mechanism is used. The aim of the players is to choose the observation more than opponent's one.

In the paper we consider the best-choice game with disorder and imperfect observation. Two players (I and II) observe sequentially $n$ iid random variables $\xi_1, ..., \xi_{\theta-1}, \xi_\theta, ..., \xi_n$ with a known continuous distribution $F_1(x)$. In random time $\theta$ the distribution of observation is changed to continuous distribution $F_2(x)$ (the disorder is happened). The moment of the cnanging the distribution has a geometric distribution, i.e. at every step the probability of disorder is $1-\alpha$. Players know parameters $\alpha$, $F_1(x)$, $F_2(x)$ but the exact moment $\theta$ is unknown.

Players may use different values of levels $x_i$ and $y_i$ (for player I and player II respectively) in every step $i \in [1, n]$. After each sampling players take a decision for acceptance or rejection of the observation. If both want to accept the same observation then a random assignment mechanism is used: player I gets the observation with probability $p$ and player II – with probability $1-p$. Each player can choose at most one observation. When some player accepts the observation at time $k$, then the other one will investigate the sequence of future realizations having an opportunity to accept one of them.

The aim of the players is to choose the observation more than opponent one. A class of suitable strategies and a gain function for the problem is constructed. The asymptotic behavior of the solution is also studied.

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# Generating Functions for Indexes of Power

Anastasiya M. Kalugina

Zabaikalsky State Humanitarian Pedagogical University
named after N.Tchernishevsky,
Chita, Russia

**Abstract**

In this work we present some models of voting game. We used generating functions for finding indexes of power. We present generating functions for Holler index, Digan-Packel index and Hoede-Bakker index.

We consider a weighted voting game with $n$ players and a quota $q$. Each player is a party with $w$ votes. A generating function can be obtained for each player. Using the generating function one can find values of Banzhaf index, Shapley-Shubik index, Holler index, Digan-Packel index. The generating functions for Banzhaf index were described by Brams and Affuso [1], and for Shapley-Shubik index - in the Cantor's work [4].

In our work we present Holler index and Deegan-Packel index in terms of generating functions.

Holler index in weighted voting game $< N, v >$ is a vector $h(v) = (h_1(v), ..., h_n(v))$, where the index of the player $i$ is equal

$$h_i(v) = \frac{m_i(v)}{\sum\limits_{i \in N} m_i(v)},\ i = 1..., n,$$

where $m_i(v)$ - number of the minimal winning coalitions, containing $i$.

Digan-Packel index in weighted voting game $< N, v >$ is a vector $dp(v) = (dp_1(v), ..., dp_n(v))$, where the index of the player $i$ is equal

$$dp_i(v) = \frac{1}{m} \sum_{S \in M : i \in S} \frac{1}{s},\ i = 1..., n,$$

where $M$ - set of all minimal winning coalitions, $m$ - general number of the minimal winning coalitions and $s$ - number of the members of a coalition $S$.

Let $< q; w_1 ..., w_n >$ - weighted voting game.

1) Number of the minimal winning coalitions $m_i(v)$ in Holler index can be presented as

$$m_i(v) = \sum_{k=q-w_i}^{q-1} \{S : \sum_{j \in S, j \neq i} w_j = k, w_{i_S} \geq k - q + w_i + 1\},$$

where $i_S : w_{i_S} = \min_{l \in S, l \neq i} w_l$, and the generating function looks like:

$$G_i(x) = \prod_{j \neq i} (1 + \gamma_i x^{w_j}).$$

2) Number of the minimal winning coalitions $m$ is equal

$$m = \sum_{k=q}^{q+w_{l_S}-1} \{S : \sum_{j \in S} w_j = k\},$$

where $l_S : w_{l_S} = \min_{l \in S} w_l$ and the generating function looks like:

$$G(x) = \prod_{i \in N} (1 + \gamma_i x^{w_i}).$$

3) Deegan-Packel index can be presented as

$$dp_i(v) = \frac{1}{m} \cdot \sum_{k=q-w_i}^{q-1} \frac{A^i(k,s)}{s+1}, \; i = 1..., n,$$

where $m$ - number of the minimal winning coalitions,
$A^i(k,s) = \{S : \sum_{j \in S, j \neq i} w_j = k, |S| = s, w_{i_S} \geq k - q + w_i + 1\}$, for which the generating function looks like:

$$G_i(x,z) = \prod_{j \neq i} (1 + z\gamma_j x^{w_j}),$$

where the symbol $\gamma_j$ is the "label" of the player $j$.

These indexes do not take into account influence of the players against each other. For that purpose Hoede-Bakker index can be used. In our work we present Hoede-Bakker index in terms of generating functions.

Let $n$ players can to either approve (accept) or reject (not accept) some decision. Let $N = \{1, ..., n\}$ be the set of all players. Assume that each player has a preference to vote "yes" (denote it by 1) or to vote "no" (denote it by 0). Let $p$ be the vector of preferences, which consists of components, 1 and 0, and specifies preferences of the players, and let $P$ be the set of all vectors of preferences. $|P| = 2^n$. The initial decision of the player is his preference. Assume that some players can influence others during the game, wherefore the final decision of the player may differ from his initial decision.

As the result, each vector of preferences $p \in P$ transforms into the vector of decision $b$, which also consists of $n$ components (0 and 1), and shows the final decisions of the players.

We apply the algorithm of Hoede and Bakker, but to the linear operator $B$.

$$b = B \cdot p,$$

where $B = (\beta_{jk})_{j,k=\overline{1,n}}$ is the matrix of influence.

$$\beta_{jk} = \begin{cases} 0, \text{if } k \text{ does not influence } j, \\ 1, \text{if } k \text{ influences } j. \end{cases}$$

$$\beta_{jj} = \begin{cases} 0, \text{if } j \text{ is under anybody's influence}, \\ 1, \text{if } j \text{ otherwise}. \end{cases}$$

Assume that the set of all players $N = \{1, ..., n\}$ can be split into 3 disjoint subsets: the set of players having influence on other players - **B**; the set of players liable to influence - **S**; the set of independent players - **I**.

Let's present Hoede-Bakker index in the terms of generating function. Then, Hoede-Bakker index is

$$HB_k = \frac{\tau_k}{2^{n-2}} - 1,$$

where $\tau_k = \sum\limits_{j=q}^{n} \alpha_j$ and $R_k^{(1)}(x) = \gamma_k x \prod\limits_{m \in N \setminus \mathbf{S}} \gamma_m^k x \prod\limits_{l \in N \setminus \mathbf{S}, l \neq k} (1 + \gamma_l x \prod\limits_{j \in \mathbf{S}} \gamma_j^l x)$.

Here, $R_k^{(1)}(x)$ - generating function for Hoede-Bakker index. The symbol $\gamma_j$ is the "label" of the player $j$. The labels have no numerical value, and fulfill the information function. The notation $\gamma_k$ means the player $k$ is not under any influence. The notation $\gamma_j^k$ means the player $k$ influences the player $j$. $1 \cdot \gamma_k = \gamma_k$, $1 \cdot \gamma_j^k = \gamma_j^k$.

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# Farsighted stability of collusive price leadership [1]

Yoshio Kamijo

Graduate School of Economics, Waseda University, 1-6-1, Nishi-Waseda, Sinjuku-ku, Tokyo 169-8050, Japan

Shigeo Muto

Department of Social Engineering, Graduate School of Decision Science and Technology, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8552, Japan

**Abstract**

The present paper studies farsighted behavior of firms to form a dominant cartel of the price leadership model by [2] and stability of the dominant cartel. Our stability concept is based on the von Neumann-Morgenstern stable set according to the indirect dominance relation. While [2] and [3] analyze the stability of *cartel size* in the price leadership model, we identify a cartel by its *constituent members*. We show that any pareto efficient and individual rational cartel is itself a stable set. Our results are mathematical extensions of the results of [13], who consider farsighted stability of $n$-person prisoner's dilemma.
JEL classification: C71, D43, L13
Keywords: price leadership model, cartel stability, foresight, stable set

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# 1 Introduction

Empirical studies have shown that in a wide variety of oligopolistic industries, there are many instances of price leadership, *i.e.*, one dominant firm or group of firms announces a new price in the industry and the remaining firms immediately follow the price. [2] This type of collusive behavior was pointed out in traditional works such as [12] and [6], and has been modeled by theoretical economists for a number of years (see, [7], [10], [4], and [14]).

The model of a finite economy with a dominant cartel and many fringe firms by [2] is one of the most seminal work in literature because they not only provide a theoretical model to predict the market but also analyze stability of the cartel. In their model, the dominant cartel acts as a leader by determining the market price, while, given the price set by the cartel, the fringe firms behave as the price taker (the fringe is called a *competitive* fringe). They also show that the stable cartel (to be precise, the stable size of the cartel) always exists if the number of firms is finite.

[3] states that stability criteria used by [2] rely on a myopic view of firms and are inconsistent with the farsighted view of the firms implicitly assumed in their model. (Details of her discussion are given in the next section.) Diamantoudi reconsiders cartel stability of the price leadership model from the viewpoint of firms' foresight and shows that the set of stable cartel sizes uniquely exists.

The purpose of this paper is along the same lines as that of [3]. That is, we analyze the stability of cartels in the price leadership model when each firm has the ability to foresee the final outcome induced by its current action. We adopt [15] stable set as our stability concept since it enables us to capture the foresight of the firm.

Although we adopt the same stability concept as that of Diamantoudi, there is a critical difference between her and our approach. d'Aspremont *et al.* as well as Diamantoudi identify a cartel by its size. In other words, two distinct cartels, which are composed of the different members, are classified into the same cartel if their sizes are equal. In contrast, we identify a cartel by its constituent members. This modification enables

[2] An example of the market where a dominant group of agents is seen as a price leader is an international oil market, where OPEC sets prices and non-OPEC members follow them.

us to suitably describe individual incentives to form a cartel.

More precisely, given a set of firms in the market $N = \{1, \ldots, n\}$, while Diamantoudi defines the dominance relation over a set of the cartel's sizes, *i.e.*, $\{0, 1, \ldots, n\}$, we define our dominance relation over $2^N := \{C : C \subseteq N\}$. The effort to capture the individual incentives in a suitable manner is successful. Although Diamantoudi shows the existence and the uniqueness of the stable set according to her dominance relation, the shape of the stable set remains open. In contrast, in our study, the stable set always exists and the shape of the stable set is revealed.

The remaining part of the paper is organized as follows. In section 2, we provide a simple example by [3], which shows that the arguments for cartel stability by [2] contains some inconsistencies and explains her way to resolve this inconsistency. Then, we illustrate that the validity of her discussion is in question if we identify a cartel by its constituent members. In section 3, we explain the model of price leadership cartel and our stability concept. Section 4 gives our results and their proofs, and we conclude in section 5.

## 2 Stabilities in the literature

In this section, we provide an example by [3] to demonstrate inconsistencies in the stability concept adopted by [2] with an implicit assumption on firms' farsighted perspective in the price leadership model. Then, we explain that Diamantoudi's approach to capture a farsighted view of firms is useful to resolve these inconsistencies. This approach, however, has a limitation since she identifies a cartel by its size. Finally, we illustrate that the validity of her discussion is in question if we identify a cartel by its constituent members.

Consider a market composed of five identical firms producing homogeneous output. Let $N = \{1, \ldots, 5\}$ denote a set of five firms. If $k$ firms form a *dominant cartel* of size $k$, the remaining firms constitute a *fringe*. When there is a size $k$ cartel, the profits of a firm in the cartel and in a fringe are denoted by $g(k)$ and $f(k)$, respectively. Table 1 shows the relationship between the profits per firm and the size of cartel for some parameters selection. [3]

[3]Here, we consider a market with a linear demand function $d(p) = 100 - p$ and

| | | |
|---|---|---|
| Perfect competition $k = 0$ | | $\swarrow f(0) = 222$ |
| $k = 1$ | $g(1) = 223$ | $\swarrow f(1) = 224$ |
| $k = 2$ | $g(2) = 226$ | $\swarrow f(2) = 230$ |
| $k = 3$ | $g(3) = 231$ | $f(3) = 241$ |
| $k = 4$ | $g(4) = 239 \nearrow$ | $f(4) = 258$ |
| Full cooperation $k = 5$ | $g(5) = 250 \nearrow$ | |

Table 1: The relationship between the profits and the cartel size (cited from [3])

Let $C_k$ denote a size $k$ cartel and $F_k$ be a fringe of cartel $C_k$. According to [2], $C_k$ is stable if no firm has an incentive to enter or exit it. That is, $C_k$ is stable if both (1) $g(k) \geqq f(k-1)$ and (2) $f(k) \geqq g(k+1)$ hold. If $k = 0$, condition (1) holds automatically, and if $k = n$, condition (2) holds.

When condition (1) is not satisfied, a firm in $C_k$ has an incentive to exit the cartel because in doing so, it can gain a profit $f(k-1)$ higher than in the current situation $g(k)$. If condition (2) does not hold, a firm in a fringe would join the cartel to gain a profit of $g(k+1)$, which is greater than its current profit $f(k)$. Thus, when either condition (1) or (2) does not hold, $C_k$ is considered as an "unstable" cartel. Meanwhile when both conditions hold, then $C_k$ is considered as a "stable" cartel.

In other words, [2] defines the following dominance relation and uses the *core* as the stability notion. A cartel $C_k$ *a-dominates* a cartel $C_h$ if either (a.1) or (a.2) holds:

**(a.1)** $k = h + 1$ and $g(k+1) > f(k)$.

**(a.2)** $k = h - 1$ and $f(k-1) > g(k)$.

A cartel $C_k$ is stable if and only if it is an element of the core, the set of the cartels which are not dominated by any other cartel according to the above dominance relation.

Following [2], $C_3$ is a unique stable cartel in the example described in Table 1. Although $C_5$ is preferred over $C_3$ by all the firms ($g(5) >$

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identical quadratic cost function of the firms $c_i(q_i) = 5q_i^2$. For details of the model, see subsection 3.1.

$f(3) > g(3)$), it is not stable since a firm in $C_5$ wants to exit and gain $f(4) = 258 > 250 = g(5)$. There is a *myopic view* behind the argument that $C_5$ is not stable because firm $i$, which decides to deviate from $C_5$, does not consider the further exits of the other firms from $C_4$ in spite of $C_4$ being dominated by $C_3$ according to the dominance relation). The important point is that when firm $i$ compares the current profit $g(5) = 250$ with $f(4) = 258$ – the one after its deviation – it is implicitly assumed that it foresees that after its deviation, the remaining cartel *readjusts* a price that suitably responds to the new circumstances, which include the four firms cartel and one fringe firm. From this viewpoint, firm $i$ has a *farsighted view*. Therefore, stability criteria based on a myopic view are inconsistent with the firm's foresight, which is implicitly assumed in the model.

Therefore we need to reconsider the stability of cartel from the viewpoint of firms' farsighted perspective. The firm's farsighted perspective is summarized as the following two type of behaviors. The first is that a firm decides to move from the current situation even if it gains less profit in the immediate aftermath of its move, when it expects that after its move, another firm, and a third firm would move and so on, and it would enjoy the more profit in the end than the current profit. The second is that a firm decides to refrain from move from the current situation even if it gains more profit immediately after its move, when it expects that after its move, a sequence of moves of firms would occur and it would obtain less profit in the end than the current situation.

[3] captures the first type of the behavior by modifying the dominance relation as follows. A cartel $C_k$ *d-dominates* a cartel $C_h, k \neq h$ if either (d.1) or (d.2) holds:

**(d.1)** If $k < h$, for any $l$ ($k < l \leqq h$), $f(k) > g(l)$ holds.

**(d.2)** If $k > h$, for any $l$ ($h \leqq l < h$), $g(k) > f(l)$ holds.

Thus, for each step of the dominance path, a firm enters (detach from) a cartel if after its move, it expects that a sequence of entries to (exits from) the cartel would occur and it compares the current profit with the profit in the final situation. The definition of the dominance relation results from the critique of [5] and the idea of [1].

The second type of the farsighted behavior is expressed by using the [15] *stable set* as the stability notion, instead of the core. The second type of the behavior means that even if $C_k$ d-dominates $C_h$, the firm that is the first mover in this dominance relation, refrains from move when there is no rationale for $C_h$ being "stable". As mentioned in [2]'s arguments, the core does not take into account the stability of the ending point of the dominance path. Following [2], $C_5$ is not stable because it is dominated by the "unstable" $C_4$, which is dominated by the stable $C_3$.

Thus, taking into account the stability of the ending point of the dominance path corresponds to the second type of the farsighted behavior of the firm. This point is well captured by the stable set. [4] The stable set is a set of outcomes satisfying two stability notions: external stability and internal stability.

Let $A$ and $\gg$ be a set of outcomes and the dominance relation defined over $A$. Then, subset $K$ of $A$ is a *stable set* if it satisfies

(i) for each $a \in A \setminus K$, there exists $b \in K$ such that $b \gg a$, and (externally stable)

(ii) for each $a \in K$, there does not exist $b \in K$ such that $b \gg a$. (internally stable)

The external stability means that any outcome outside the stable set is attracted into the set, and thus, the outcome does not "prevail" in this society. On the other hand, the internal stability gives the rationale to this attraction such that the end point of this attraction is also "stable". When the dominance relation is extended to the farsighted version, even if an individual (or a group of individuals) deviates from an outcome $a$ in the stable set and can induce outcome $b$, $b$ is dominated by outcome $c$ in the stable set (the external stability), and thus, he refrains from the deviation since $c$ is not profitable to $a$ for him (the internal stability).

[3] shows that the stable set according to the dominance relation is $\{C_3, C_5\}$. A cartel $C_5$ is considered to be stable since $C_4$, which d-dominates $C_5$, is d-dominated by $C_3$, which is also considered as stable cartel and $C_3$ does not d-dominated $C_5$. On the other hand, $C_3$ d-dominates $C_0$, $C_1$, $C_2$ and $C_4$, and $C_5$ does not d-dominates $C_3$. Thus, $\{C_3, C_5\}$ is a stable set.

Although [3]'s discussion mentioned above is persuasive, there is an inadequacy in her discussion. To see this, let reconsider the stability of

[4] Another way to capture this point is to use the *credible core* by [9].

a cartel $C_3$. The stability that the stable set assumes is that even if the deviation from an outcome in the stable set to an outcome outside the set exists, the outcome outside the set is attracted into other outcome in the set (externally stable) and the first deviant firm never gain from this sequence of deviation (internally stable), and thus farsighted individual refrains from the first deviation. Is this appropriate to stability of $C_3$? If a firm detaches from $C_3$, $C_2$, a cartel after its exit, is dominated by $C_3$ and thus, $C_3$ prevails. It appears true, but actually doubtful because there is no reason that the first deviant firm from $C_3$ to $C_2$ and the second from $C_2$ to $C_3$ is the same firm.

Now we identify cartel by its members and let $C_3 = \{1, 2, 3\}$ and $C'_3 = \{2, 3, 4\}$ denote two distinct size 3 cartels. Consider a path from $C_3$ to $C'_3$ such that first firm 1 exits and then firm 4 in a fringe joins the cartel. Since firm 1 prefers the profit $f(3) = 241$ in $C'_3$ to $g(3) = 231$, and firm 4 prefers the profit $g(3) = 231$ in $C'_3$ to the profit $f(2) = 230$, which firm 4 obtains when there is a cartel $C_2 = \{2, 3\}$, a dominance path from $C_3$ to $C'_3$ seems to exist.

$$C_3 = \{1, 2, 3\} \xrightarrow{\text{1's exit}} C_2 = \{2, 3\} \xrightarrow{\text{4's entry}} C'_3 = \{2, 3, 4\}.$$

This inadequacy is due to the fact that [3] considers the stability of *cartel sizes*, not cartel. In other words, [3] (as well as [2]) identifies cartel by its size and two distinct cartels with equal size are considered as the same one. Since the dominance relation is derived by the individual incentives to deviate, we have to pay attention to who forms a cartel and who belongs to a fringe. Thus, it seems natural that a cartel is identified by its *constituent members.*

Therefore, in this paper, we define dominance relation with respect to firms' farsighted perspective over the set of cartels that are identified by their members, *i.e.*, $\{C : C \subseteq N\} = 2^N$ given the set of firms $N$, and analyze stability of a dominant price leadership cartel. Another difference between our and Diamantoudi's approach is that we allow a coalitional move following many existing studies.

In these settings, we show the following result:

*If a cartel $C \subseteq N$ is pareto efficient and individual rational, then $C$ is a stable cartel. That is, $\{C\}$ is a stable set according to our dominance relation.*

Thus, we uncover the shapes of the stable sets. In the next section, we explain the price leadership model of [2] and provide the definition of our stability concept.

# 3 The model

## 3.1 *Collusive price leadership*

We consider an industry composed of $n$ ($n \geqq 2$) identical firms, which produce a homogeneous output. If $k$ firms decide to form a cartel and set a price $p$, the remaining $n-k$ firms constitute a competitive fringe and decide each output $q_f(p)$ by

$$p = c'(q_f(p)),$$

where $c'$ is firm $i$'s marginal cost, which satisfies $c' > 0$ and $c'' > 0$.

Let $d(p)$ be a market demand function satisfying $d' < 0$. Members of a dominant cartel choose the price that maximize their joint profit, given the supply decision of a competitive fringe. Since the marginal cost is increasing, the maximization of the joint profit is achieved by equal division of their total output. Therefore, each firm of a cartel behaves as a monopolist with respect to the individual residual demand function defined as $\frac{d(p)-(n-k)q_f(p)}{k}$. Thus, the price that a cartel actually chooses is obtained as follows:

$$Max_{p>0} \, \frac{d(p)-(n-k)q_f(p)}{k} \, p - c\left(\frac{d(p)-(n-k)q_f(p)}{k}\right)$$

According to the price as a solution of the above problem, the profits of a cartel firm and a fringe firm are obtained for each $k$ ($k = 1, \ldots, n$) and are denoted by $g(k)$ and $f(k)$, respectively. If $k = 0$, that is, there is no cartel, then it is assumed that the market structure is competitive. Therefore, $f(0)$ is defined by a profit of a fringe firm for a competitive price $p^{comp}$, which satisfies $d(p^{comp}) = nq_f(p^{comp})$.

In this setting, the following proposition is shown by [2].

**Proposition 1 ([2])** *The following properties about the profits of a cartel firm and a fringe firm hold.*

*(i)* $f(k) > g(k)$ *for each* $k$ $(= 1, \ldots, n-1)$.

*(ii)* $g(k)$ *is an increasing function in* $k$.

This proposition says that there is a dilemma structure in firms who decide to join or detach from a cartel. The second property of Proposition 1 means that a firm who belongs to a cartel prefers the cartel being larger. The first property indicates that it is more profitable for each firm to belong to a fringe if the size of a cartel is unchanged.

This dilemma structure is similar to but weaker than the typical *prisoner's dilemma* situation. In the prisoner's dilemma, the "defect" is more profitable than the "cooperate", irrespective of the others choices. When we interpret cooperate and defect as joining and not joining a cartel, $f(k-1) > g(k)$ always holds in the prisoner's dilemma. In contrast to the prisoner's dilemma, in the price leadership model, each cartel firm wants to switch positions with a fringe firm ($f(k) > g(k)$). Thus, cartel firm envies a fringe firm's position.

The next proposition shows that fringe firms prefer a situation with a dominant cartel to one without it.

**Proposition 2** *For any* $k$ $(= 1, \ldots, n-1)$, $f(0) < f(k)$ *holds.*

**Proof.** Suppose that there exists a $k$ firms cartel. If the cartel chooses price $p = p^{comp}$, then they can gain $f(0)$, the profit for a competitive equilibrium. Since they set a price to maximize their profit, $g(k) \geqq f(0)$ holds. This inequality and (i) of Proposition 1 implies $f(k) > f(0)$. □

In the rest of the paper, we analyze the stability of the price leadership cartel characterized by profit functions $f$ and $g$, which satisfy the properties described in Propositions 1 and 2. In order to expand the scope of our discussion, we establish our discussion and complete proof of our theorems in a weaker setting than the conditions mentioned above. We impose the following three conditions on functions $f$ and $g$.

**Assumption 1**: $f(0) \leqq f(k)$ for any $k = 1, \ldots, n-1$.
**Assumption 2**: $g(k)$ is an increasing function in $k$.
**Assumption 3**: $g(n) > f(0)$.

Thus, we provide our discussion without mentioning a dilemma structure ($f(k) > g(k)$) in the price leadership model. Assumption 3 is obtained from the facts that $g(k)$ is an increasing function and $g(k) \geqq f(0)$ holds by the proof of Proposition 2.

### 3.2 *Stability*

We begin by explaining some additional notations and definitions to express our stability concept. Let $N = \{1, \ldots, n\}$ and $X = \times_{i \in N} X_i = \{0, 1\}^n$ be a set of firms and a set of cartels, respectively. For each $i \in N$, $x_i = 1$ implies that firm $i$ belongs to a dominant cartel, and $x_i = 0$ implies that firm $i$ belongs to a competitive fringe. An element $x \in X$ is called a *cartel structure*, or simply a *cartel* since each $x \in X$ corresponds to a unique cartel $C \subset N$. For each $x \in X$, the set of firms that belong to a cartel is denoted by $C(x) = \{i \in N : \, x_i = 1\}$, and the set of fringe firms is denoted by $F(x) = N \setminus C(x)$.

The payoff function $u_i : X \to \mathbb{R}$ for each $i \in N$ is defined as follows:

$$u_i(x) = \begin{cases} g(k) & \text{if } x_i = 1, \\ f(k) & \text{if } x_i = 0, \end{cases}$$

where $|C(x)| = k$. Functions $f$ and $g$ satisfy Assumptions 1, 2, and 3.

Let $x \in X$ and $y \in X$ be two distinct cartel structures. We say that a cartel structure $x$ *pareto dominates* $y$, and denote $xPy$ if $u_i(x) \geqq u_i(y)$ holds for all $i \in N$ and strict inequality holds for some $j \in N$. If $x$ is not pareto dominated by any other cartel, the $x$ is called a *pareto efficient* cartel structure. The set of all the pareto efficient cartel structures is denoted by $X^P \subseteq X$. Since $x = (1, \ldots, 1)$, that is, the grand cartel is pareto efficient by Assumptions 2 and 3, $X^P$ is not empty.

We define $X^* \subseteq X$ by

$$X^* = \{x \in X : \, g(|C(x)|) > f(0)\}.$$

Since $g$ is increasing by Assumption 2 and $g(n) > f(0)$ holds by Assumption 3, there exists integer $s^* \in \mathbb{N}$ such that $X^* = \{x \in X : \, |C(x)| \geqq s^*\}$. Let $v_i$ be a minimax payoff of firm $i \in N$, *i.e.*, $v_i = \min_{x_{-i}} \max_{x_i} u_i(x)$. Then,

$$v_i = \max\{g(1), f(0)\} \tag{1}$$

by Assumptions 1 and 2. If cartel structure $x$ satisfies $u_i(x) \geqq v_i$ for all $i \in N$, then $x$ is called an *individual rational* cartel structure. If strict equality holds for all $i \in N$, then $x$ is called a *strictly individual rational* cartel structure. We denote the sets of all the individual rational cartel structures and the strictly individual rational cartel structures by $X^I$ and $X^{SI}$ respectively. By its definition, $X^{SI} \subseteq X^*$.

A cartel structure $y$ is *inducible* from $x$ through coalition $S \subseteq N$ if $x_i = y_i$ holds for any $i \in N \setminus S$ and we write $x \to_S y$. Clearly, if $x \to_S y$ holds, then $y \to_S x$ also holds. For a coalition $S \subseteq N$, we write $x \succ_S y$ if $u_i(x) > u_i(y)$ holds for any $i \in S$.

Next, let us define a dominance relation over $X$. Since there exists one-to-one correspondence between $X$ and $2^N$, it is equivalent to define a dominance relation over a set of cartels. Following [5] and [1]'s perspective, we define *indirect dominance relation* over $X$ that captures the ability for each firm to foresee the final outcome which is induced by the firm's current behavior.

**Definition 1** *A cartel structure $x$ is indirectly dominated by $y$ and we write $y \gg x$ if there exist a (finite) sequence of cartels $x^0, x^1, \ldots, x^M$ with $x^0 = x$ and $x^M = y$ and a sequence of coalitions $S^1, \ldots, S^M$ such that for each $m\ (= 1, \ldots, M)$ (a) $x^{m-1} \to_{S^m} x^m$ and (b) $y = x^M \succ_{S^m} x^{m-1}$.*

If $M = 1$, we say that $x$ is directly dominated by $y$ via coalition $S^1$.

A farsighted stable set is a stable set or [15] solution defined by dominance relation $\gg$ over $X$. Formally, a subset $K$ of $X$ is called a **farsighted stable set (FSS)** if the following conditions hold:

(i) For any $x \in K$, there does not exist $y \in K$ such that $y \gg x$ (internal stability of $K$).

(ii) For any $z \in N \setminus K$, there exists $x \in K$ such that $x \gg z$ (external stability of $K$).

In the next section, we characterize FSSs of the price leadership cartels.

# 4 Results

In this section, we reveal the complete shapes of farsighted stable sets for a price leadership model. Our main statement is that a pareto efficient and individual rational cartel is itself a farsighted stable set and there is no other type of farsighted stable set except for some degenerate cases.

First we prove the following two lemmas on the properties of pareto efficient cartels and individual rational cartels. Let $x^c \in X$ and $x^f \in X$ denote $(1, \ldots, 1)$ and $(0, \ldots, 0)$, respectively. That is, $x^c$ represents the grand cartel structure and $x^f$ represents a competitive situation.

**Lemma 1** *The following properties hold:*

**(i)** *A cartel structure $x \in X, x \neq x^c$ is pareto efficient, that is, $x \in X^P \setminus \{x^c\}$ if and only if $f(|C(x)|) > g(n)$.*

**(ii)** *$x \in X^P \setminus \{x^c\}$ if and only if $x \gg x^c$.*

**(iii)** *For any $x \in X^P$ and for any $y \in X$ with $y \neq x$ and $y \neq x^f$, there exists $i \in C(y)$ such that $u_i(x) > u_i(y)$.*

**Proof.** (i) Suppose that $x$ satisfies $f(|C(x)|) > g(n)$ and that there exists $y \in X$ such that $yPx$. For $i \in F(x)$, $y_i$ must be '0' since $u_i(x) = f(|C(x)|) > g(n) \geqq g(|C(y)|)$ by Assumption 2. Thus, $C(x) \supsetneq C(y)$. For $i \in C(y) \subsetneq C(x)$, $u_i(y) = g(|C(y)|) < g(|C(x)|) = u_i(x)$. This contradicts $yPx$.

Next we show the only if part. Suppose that $x \in X^P$, $x \neq x^c$ but $f(|C(x)|) \leqq g(n)$. Then $x^c$ pareto dominates $x$ by Assumption 2 and this contradicts $x \in X^P$.

(ii) We first show the 'if' part. If $x \gg x^c$ holds, then there exists the first deviant coalition $S$ from $x^c$ to the final outcome $x$. Hence, $x \succ_S x^c$ holds. Let $i \in S$. Then, $x_i = 0$ because there is no cartel better for a cartel firm than grand cartel structure $x^c$. The fact that $u_i(x) > u_i(x^c)$ implies that $f(|C(x)|) > g(n)$. Thus, $x$ is pareto efficient by (i) of this lemma.

Next, we show the "only if" part. Suppose $x \in X^P \setminus \{x^c\}$. Then, by (i) of this lemma, $u_i(x) > u_i(x^c)$ for any $i \in F(x)$. Thus, $x$ directly dominates $x^c$ via $F(x)$.

(iii) If $x_i = 1$ for any $i \in C(y)$, then $C(x) \supsetneq C(y)$ since $x \neq y$. Therefore, $|C(x)| > |C(y)|$. This implies that for any $i \in C(y)$, $u_i(x) = g(|C(x)|) > g(|C(y)|) = u_i(y)$. Otherwise, there exists $i \in C(y)$ such that $x_i = 0$. Then $u_i(x) = f(|C(x)|) > g(n) \geqq g(|C(y)|) = u_i(y)$. The second strict inequality is by (i) of this lemma and the third inequality is by Assumption 2. □

**Lemma 2** *The following properties hold:*

**(i)** *$x \in X^*$ if and only if $x \gg x^f$.*

**(ii)** *Let $x \in X \setminus \{x^f\}$. Then, $x^f \gg x$ if and only if $f(0) > g(|C(x)|)$.*

**Proof.** (i) First we show the "only if" part. Let $x \in X^*$. Then, $x$ directly dominates $x^f$ via coalition $C(x)$ because all the firms in $C(x)$ prefer $x$ to $x^f$ by the definition of $X^*$ and $x$ is inducible from $x^f$ through coalition $C(x)$.

Next, we show the "if" part. Suppose $x$ indirectly dominates $x^f$. By the definition of the indirect dominance relation, there exist a sequence of the cartel structures $x^f = x^0, x^1, \ldots, x^M = x$ and a sequence of the deviant coalitions $S^1, \ldots, S^M$ satisfying conditions (a) and (b) in Definition 1. Let $k$ be the first natural number such that there exists $i \in C(x)$ with $x_i^k = 0$ and $x_i^{k+1} = 1$. Then, $g(|C(x)|) > f(|C(x^k)|) \geqq f(0)$ by Assumption 1.

(ii) Suppose $f(0) > g(|C(x)|)$. Then, all the firms in $C(x)$ prefers $x^f$ to $x$ and $x^f$ is inducible from $x$ through $C(x)$. Thus, $x^f$ directly dominates $x$ via $C(x)$.

Next, suppose $x^f \gg x$. Let $S \subseteq N$ be the first deviant coalition in the dominance relation from $x$ to $x^f$. If there exists $i \in S$ with $x_i = 0$, then $f(0) > f(|C(x)|)$ by the definition of the dominance relation and this contradicts Assumption 1. If there exists $i \in S$ with $x_i = 1$, then $f(0) > g(|C(x)|)$. □

The first and the second theorems show that any pareto efficient and individual rational cartel structure is a FSS.

**Theorem 1** *For any $x \in X^* \cap X^P$, $\{x\}$ is a farsighted stable set.*

**Proof.** Since $\{x\}$ consists of one point, we only consider the external stability. Take any $y \in X$, $y \neq x$. When $y = x^f$, $x$ dominates $y$ by (i) of Lemma 2.

When $y \neq x^f$, by (iii) of Lemma 1, there exists $i_1 \in C(y)$ such that $u_{i_1}(x) > u_{i_1}(y)$. Then, we construct a cartel structure $y^1$ as follows. For all $i \in N$,

$$y_i^1 = \begin{cases} 0 & \text{if } i = i_1, \\ y_i & \text{if } i \neq i_1. \end{cases}$$

If either $y^1 = x$ or $y^1 = x^f$ holds, then we stop this process. Otherwise, there exists $i_2 \in C(y_1)$ such that $u_{i_2}(x) > u_{i_2}(y^1)$. Then, we construct $y^2$ by

$$y_i^2 = \begin{cases} 0 & \text{if } i = i_2, \\ y_i^1 & \text{if } i \neq i_2. \end{cases}$$

If $y^2 = x$ or $y^2 = x^f$, then we stop this process. Otherwise, there exists $i_3 \in C(y^2)$ such that $u_{i_3}(x) > u_{i_3}(y^2)$ and we continue the same process at most $n-3$ times.

Since $N$ is finite, we can find an integer $M \in \mathbb{N}$, a sequence of players $i_1, \ldots, i_M$, and a sequence of outcomes $y = y^0, y^1, \ldots, y^M$ such that $y^M$ is either $x$ or $x^f$, $y^{m-1} \rightarrow_{\{i_m\}} y^m$ for any $m = 1, \ldots, M$ and $x \succ_{\{i_m\}} y^{m-1}$ for any $m = 1, \ldots, M$. If $y^m = x$, then $x$ indirectly dominates $y$. If $y^m = x^f$, then $x$ dominates $y^m$ by (i) of Lemma 2, and therefore, $x$ indirectly dominates $y$. We obtain the desired result. □

**Theorem 2** *If $g(1) \leqq f(0)$, $X^* \cap X^P = X^{SI} \cap X^P$. Otherwise, $X^* \cap X^P = X^I \cap X^P$.*

**Proof.** If $g(1) \leqq f(0)$, $v_i = f(0)$ by equation (1). Then, $x \in X^P \cap X^*$ is strictly individual rational since if $i \in F(x)$, then $u_i(x) = f(|C(x)|) > g(n) > f(0)$ by Lemma 1 and if $i \in C(x)$, then $u_i(x) = g(|C(x)|) > f(0)$ by Lemma 2. Since $X^{SI} \subseteq X^*$, $X^* \cap X^P = X^{SI} \cap X^P$ holds.

If $g(1) > f(0)$, then $X^* = X \setminus \{x^f\}$. By Assumption 3, $X^* \cap X^P = X^P$. Let $x \in X^P$. Then, $x \neq x^f$. For $i \in C(x)$, $g(|C(x)|) \geqq g(1)$ by Assumption 2. For $i \in F(x)$, $f(|C(x)|) > g(n) > g(1)$ since $x \in X^P$. Therefore, $X^P \subseteq X^I$ and $X^I \cap X^P = X^P$. □

Theorem 1 and Theorem 2 say that the cartel structure that is pareto efficient and (strictly) individual rational is a FSS.

**Lemma 3** *If there does not exist an integer $k^*$ such that $f(0) = g(k^*)$, the fact $x \in X^*$ and $y \notin X^*$ means that $x$ dominates $y$.*

**Proof.** Because of the assumption of this lemma, $y \notin X^*$ means that $g(|C(y)|) < f(0)$. For any $i \in C(y)$, if $x_i$ is 1, then $u_i(x) = g(|C(x)|) > f(0) > g(|C(y)|)$ since $x \in X^*$. If $x_i$ is 0, then $u_i(x) = f(|C(x)|) \geqq f(0) > g(|C(y)|)$ by Assumption 1. Thus, all the firms in $C(y)$ prefers $x$ to $y$. Moreover, all the firms in $C(x)$ prefer $x$ to $x^f$ by the definition of $X^*$. Therefore, the deviation defined by (i) first, all the members in $C(y)$ detaching from the cartel and (ii) next, all the members in $C(x)$ forming a cartel, implies that $x$ indirectly dominates $y$. □

Next theorem shows that without some degenerate cases, there is no farsighted stable set other than the ones defined in Theorem 1.

**Theorem 3** *If either condition (1) or (2) holds, then there exists no farsighted stable set other than that given in Theorem 1:*

**(1)** $X^P \subseteq X^*$.

**(2)** *There does not exist an integer* $k^*$ *such that* $f(0) = g(k^*)$.

**Proof.** Take any farsighted stable set $K$ that is different from the set given in Theorem 1. Then, $K$ does not contain any outcome that belongs to $X^* \cap X^P$ since, as shown in Theorem 1, such a cartel structure indirectly dominates all the others and this contradicts the internal stability of $K$.

Suppose condition (1) is satisfied. If $x \in K$, then $x \notin X^P$ since otherwise, $x$ is an element of $X^*$ by the supposition. Thus $K \cap X^P = \emptyset$ and this implies that by (ii) of Lemma 1 there exists no $x \in K$ such that $x \gg x^c$. Since $x^c \in X^P \cap X^*$ and thus $x^c \notin K$, the external stability of $K$ does not hold.

Suppose condition (2) holds. If $K \cap X^* \neq \emptyset$, then $K$ does not have an element in $X \setminus X^*$ because otherwise the internal stability of $K$ does not hold by Lemma 3. Of course, $K$ does not have an element in $X^* \cap X^P$ because of the argument of the first paragraph in this proof. Thus, $K \subseteq X^* \setminus X^P$. However, any element in $K$ does not indirectly dominate $x^c$ by (i) of Lemma 1 and this contradicts the external stability of $K$.

When $K \cap X^* = \emptyset$, for any $x \in K$, $g(|C(x)|) < f(0)$ holds by Assumption 2 and condition (2). Since $K$ is not a set $\{x^f\}$, we can take $x \in K$ with $x \neq x^f$. Then, $x$ is indirectly dominated by $x^f$ because of (ii) of Lemma 2. To guarantee the internal stability of $K$, $K$ does not have $x^f$. In order to preserve the external stability of $K$, some element $y \in K$ must indirectly dominate $x^f$. However $y$ must be an element in $X^*$ by (i) of Lemma 2 and this is a contradiction. □

In the next theorem, we describe the shapes of FSSs in the degenerate cases, *i.e.*, $f(0) = g(k^*)$ for some $k^*$, using an additional assumption. The assumption in the next theorem is, however, satisfied in the price leadership model (Proposition 2).

**Theorem 4** *Assume that* $f(0) < f(k)$ *for any* $k(= 1, \ldots, n-1)$. *Consider the case where there exists an integer* $k^*$ *such that* $f(0) = g(k^*)$. *If the following condition (a) holds,*

$$f(k^*) > g(n),$$

*then* $K_1 = \{x^f\} \cup \{x \in X : |C(x)| = k^*\}$ *is a farsighted stable set in addition to those given in Theorem 1 and there is no farsighted stable set. Otherwise there is no farsighted stable set except for ones given in Theorem 1.*

**Proof.** First, we show that $K_1$ is a farsighted stable set when condition (a) holds. Let $K_2 = K_1 \setminus \{x^f\}$. By (i) of Lemma 2, any element in $K_2$ does not indirectly dominate $x^f$, and by (ii) of Lemma 2, $x^f$ does not indirectly dominate any element in $K_2$. Let $x \in K_2$, $y \in K_2$, $x \neq y$. For $x$ to indirectly dominate $y$, there exist some element $x' \in X$ and some firm $i \in N$ such that firm $i$ prefers $x$ to $x'$, and $x_i = 1$ and $x'_i = 0$. However, there does not exist such $x'$ because $f(|C(x')|) \geqq f(0) = g(|c(x)|)$ by Assumption 1. Thus, the internal stability of $K_1$ holds.

Let $y \in X$ such that $|C(y)| > k^*$. Take any $S \subseteq C(y)$ such that $|S| = |C(y)| - k^*$. Then, a cartel structure $x$ such that $x_i = 1$ for all $i \in C(y) \setminus S$ and $x_i = 0$ otherwise, directly dominates $y$ via $S$ because in cartel structure $x$, all the firms in $S$ are in a fringe and they prefer $x$ to $y$ $(f(k^*) > g(n) > g(k))$ by condition (a) and Assumption 2. Next, let $y \in X, y \neq x^f$ such that $|C(y)| < k^*$. Then, $x^f$ dominates $y$ by (ii) of Lemma 2. Hence, the external stability of $K_1$ holds, and thus, $K_1$ is a farsighted stable set.

Let $K$ be a farsighted stable set other than those in Theorem 1. If $x^f \notin K$, there exists $x \in K$ such that $x$ dominates $x^f$. By (ii) of Lemma 2, $x \in X^*$ and $|C(x)| \geqq k^* + 1$. Then, $x \notin X^P$ since otherwise $\{x\}$ becomes a farsighted stable set described in Theorem 1. By (ii) of Lemma 1, $x$ does not dominate $x^c$. Thus, there exists $y \in K$ such that $y$ dominates $x^c$, and thus, $y \in X^P$ by (ii) of Lemma 1. To preserve the internal stability of $K$, $y \in X^P \setminus X^*$. Thus, $|C(y)| \leqq k^*$. Then, $g(|C(y)|) \leqq f(0) < f(|C(x)|)$ and $f(0) < g(|C(x)|)$ hold by the assumption of this theorem and the definitions of $X^*$ and $X^P$. Hence, $x$ indirectly dominates $y$ through $y \rightarrow_{C(y)} x^f \rightarrow_{C(x)} x$ because all the firms in $C(y)$ prefer $x$ to $y$ and all the firms in $C(x)$ prefers $x$ to $x^f$. This contradicts the internal stability of $K$.

Consider the case that $x^f \in K$. Clearly $\{x^f\} \neq K$. Thus, there exists $x \in K$ with $x \neq x^f$. To preserve the internal stability, $x$ satisfies $|C(x)| = k^*$ since by Lemma 2, $x$ dominates $x^f$ if $|C(x)| > k^*$ and $x^f$ dominates $x$ otherwise. Moreover, as shown in the first paragraph of this proof, $x$ does not indirectly dominate $y$ such that $|C(y)| = k^*$. To preserve the external stability of $K$, $K$ must be $\{x^f\} \cup \{x \in X : |C(x)| = k^*\}$. As shown, this $K$ is in fact a farsighted stable set if $f(k^*) > g(n)$. Otherwise, $K$ is not a farsighted stable set because $x^c$ pareto dominates all the elements in $K$. $\square$

In this paper, it is assumed that to induce a cartel structure $x$ from another cartel structure $y$, it is enough that all the firms that actually move (*i.e.*, enter or exit from a cartel) agree to this movement. From a viewpoint of coalition formation theory, however, it is often assumed that the permission of the members in a current cartel is necessary for a fringe firm to join the cartel. Meanwhile, firms in the cartel can exit from the cartel in a unilateral way. As a result, we can redefine the inducement relations as follows:

$$x \rightarrow_S y \iff x_i = y_i \, \forall i \in N \setminus S, \text{ and}$$
$$\text{if } C(y) \setminus C(x) \neq \emptyset, \text{ then } S \supseteq C(x) \cap C(y).$$

This definition reflects the fact that on one hand, players in $C(x) \setminus C(y)$ detach from the cartel in a unilateral way, and on the other hand, to join the cartel, players in $C(y) \setminus C(x)$ need the permission of members in $C(x) \cap C(y)$.

It is possible to redefine the indirect dominance relation and the farsighted stable set according to the above inducement relations. An important point is that this restriction on inducement relations does not alter our conclusions. That is, Theorems 1, 2, 3, and 4 hold when this new indirect dominance relation is used. The reason is that for every indirect dominance path that is used in the lemmas and the theorems in this paper, either members in a cartel prefer the final cartel structure to the current situation or there is no cartel whenever fringe firms form a cartel.

# 5 Conclusions

In this paper, we analyzed the stability of a dominant cartel model of the price leadership introduced by [2]. The solution concept adopted in this study is the stable set with indirect dominations, which reflect firms' farsighted view.

The dominance relation proposed by [3] is defined over a set of cartel sizes, *i.e.*, $\{0, \dots, n\}$. Thus, in her paper, two distinct cartels are considered as the same when these sizes are equal. In contrast, we distinguish one cartel from the other when the constituent members are different even if they are of the same size. Hence, we define our dominance relation over $\{C : C \subseteq N\}$. Another difference between our and [3]'s approach is that while she considers an individual move for each step of the dominance path, we allow a coalitional move following many other literatures.

We can find the complete shapes of farsighted stable sets, which remain open in [3]. Our results imply the possibility of cooperation in the dilemma situation and shed some light on dissolution of the dilemma, which has been widely studied by non-cooperative approach and equilibrium concept. In addition, our discussions have policy implications on a market structure because our theorem (Theorem 1) shows that there is the possibility of firms forming a large cartel even if the decision problem of the firms joining or not joining a cartel is in a dilemma situation. These implications are in contrast to the results of [11] and [8] who analyze cartel formation in non-cooperative way and show that firms encounter some difficulty to form a large cartel.

Finally, our theorems are mathematical extensions of the results of [13] who study farsighted stability of the $n$-person prisoner's dilemma described by a normal form game and show that a pareto efficient and individual rational outcome is itself a farsighted stable set.

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# N-Player Game in a Multiple Access Channel is Selfish

Andrey Lukyanenko

Helsinki Institute for Information Technology, Helsinki, Finland

Igor Falko

Institute of Applied Mathematical Research of KarRC RAS, Petrozavodsk, Russia

Andrei Gurtov

Helsinki Institute for Information Technology, Helsinki, Finland

**Abstract**

This paper studies behavior of players in a common exclusively-shared channel using a backoff protocol for resolving collisions. We show that when players have freedom to choose backoff parameters (or time to send a next packet), they behave selfishly. The system has an undesirable Nash equilibrium, where every player tries to grasp as much channel as possible. Since the channel is exclusively shared, no player would get a packet through (all packets will collide). Although the result is seemingly obvious, we were unable to find it in the literature. We also evaluate a simple incentive mechanism based on an arbiter model, which controls channel access by jamming misbehaving players.

## 1 Introduction

The backoff protocol is a scheduling protocol for simultaneous access to a multiple access channel where simultaneous transmissions collide. To deal with collisions, a backoff protocol was introduced and adopted in such protocols as Aloha [2], Ethernet [13] and IEEE 802.11 (Wi-Fi) [1]. As an example, Aloha protocol uses a constant backoff protocol, while IEEE 802.11 uses a truncated exponential binary backoff protocol.

Over past thirty years, the backoff protocol was analyzed by several researchers [3, 7, 8, 5, 6, 4]. Furthermore, following the idea by Kwak et al. [11] we analyzed general backoff protocols [12]. We studied optimality of a general backoff function instead of a fixed function. The analysis showed that the choice of the optimal protocol parameters depends on the number of active stations in the network and may vary depending on the load of the network. Hence, permitting the stations to choose the backoff parameters depending on the channel load can increase throughput for individual stations and the network itself.

On the other hand, recent studies on game-theoretic aspects of the backoff protocols showed that the freedom to control backoff parameters leads to selfish behavior of individual players (stations) [10].

In this paper we consider what if we give freedom to manipulate general backoff parameters to each station in the network. In other words, if a station is free to use the channel at any time, what the resulting behavior would be?

Unlike in the backoff model, here we do not give the history of interaction to a station. Hence, the network model is a black box to the end station. A station does not know had the packet collided before the game is finished, stations know only the number of other stations (players) and that every player in the network wants to selfishly maximize its throughput. Unfortunately, we omit consideration of the previous history (backoff counter) because it makes the model very complex otherwise. We believe that the model still represents the choice of each player as with a general backoff network without restrictions on behavior. Under these conditions, we show that the game has undesirable Nash equilibrium.

Additionally, we modify the model using a known incentive mechanism — a common network arbiter, which jams the channel if some player transmits too much packets. We show that these incentives do not give a unique Nash equilibrium solution, and one of the possible equilibrium solutions still involves undesirable behavior.

# 2 Analysis

## 2.1 Model

Consider the following game model. $N$ players try to send packets through a shared channel during time $T$. The whole time is divided into timeslots; the length of each timeslot is 1, hence there are $T$ slots of length 1, which are synchronized and known to each player. During one timeslot player can send one packet. At the beginning of each game, every player chooses timeslots for sending packets. We assume that a player $i$ decides to use $k_i$ slots for transmission. Knowing the number of slots to be used, the exact slots for transmission are chosen randomly and uniformly among other possible. There are $\binom{T}{k_i}$ combinations to place $k_i$ elements on $T$ and probability for every combination is equal. For such a game we want to find which strategy (a number of packets to send) a player will choose.

A similar problem was studied by Kolchin et al. in the book "Random allocations" [9]. The difference is that the book did not consider a game problem, but used the same $k_i$ for every player. Even for such problem, it is hard to analyze the collision probability. In our case, the probability that $k_1$ and $k_2$ will collide exactly in $\Delta$ slots equals to $\frac{\binom{k_1}{\Delta}\binom{T-k_1}{k_2-\Delta}}{\binom{T}{k2}}$.

## 2.2 Two-player game

Consider a particular case of the game above, when the number of players is two. The first player decides to send packets in $k_1$ slots, the second in $k_2$ slots. As in [9] consider the following random variable $\mu_r$ be the number of slots, during which $r$ packets are sent ($0 \leq r \leq N$). In case of two-player game, there are at most two packets in a slot from both players. Now, let us calculate $\mu_2$. If we define as $q_i$ the event that two packets were sent in slot $i$, then $\mu_2 = \sum_{i=1}^{T} \mathbb{1}\{q_i\}$, where $\mathbb{1}\{A\}$ is an indicator function for event $A$. Taking expectation from the equation we get $E\mu_2 = TP\{q_i\}$, and for a two-player game it is equal to $E\mu_2 = T\frac{k_1}{T}\frac{k_2}{T} = \frac{k_1 k_2}{T}$.

That value is exactly the expected number of collided packets. The expected number of successful packets for the first player is $k_1 - \frac{k_1 k_2}{T}$ and for the second player is $k_2 - \frac{k_1 k_2}{T}$. Hence, we have the income function $H_1(k1, k2) = k_1(1 - \frac{k_2}{T})$ for the first player and $H_2(k1, k2) = k_2(1 - \frac{k_2}{T})$ for the second. It is clear that unless one of players chooses $T$ as a strategy, the best income for another player is to choose $T$ as a strategy. This is a Nash equilibrium. Since we assumed that players behave similarly, we can assume that the Nash equilibrium is $(T, T)$. Each of two players behaves selfishly.

## 2.3 N-player game

Now, consider an N-player game. It can be reduced to a two-player game, if we consider the first player as one player, and the rest of players as another player. Hence, if we define $\Delta$ as the number of slots taken by the rest of the players, then the income for the first player will be

$$H_1(k_1, k_2, \ldots, k_N) =$$

$$\sum_{\Delta=0}^{T} k_1(1 - \frac{\Delta}{T})P\{k_2, \ldots, k_N \text{ occupies } \Delta\} = k_1(1 - \frac{E\Delta}{T}).$$

Consider again, $\mu_r$. Now we need to find $\mu_0$, the number of free slots for players $\overline{2,N}$. Let $q_i$ be an event that slot $i$ is unoccupied by players $\overline{2,N}$. Then $\mu_0 = \sum_{i=1}^{T} \mathbb{1}\{q_i\}$. The expectation of this value is $E\mu_0 = TP\{q_i\} = T\prod_{i=2}^{N}(1 - \frac{k_i}{T})$. Thus, the expected number of free slots is $T(1 - \prod_{i=2}^{N}(1 - \frac{k_i}{T}))$, and hence the income function for the first player is $H_1(k_1, k_2, \ldots, k_N) = k_1\prod_{i=2}^{N}(1 - \frac{k_i}{T})$. The income for player $j$ is

$$H_j(k_1, k_2, \ldots, k_N) = k_j \prod_{i=1, i\neq j}^{N} (1 - \frac{k_i}{T}).$$

From here, we again see that unless one of the other players chooses $T$ as a strategy, any player is forced to choose $T$. Because of similarity and as players cannot know what other players choose, the expected Nash equilibrium for the game will be $(T, \ldots, T)$. Hence, the N-player game leads to selfish behavior.

## 2.4 On optimality and improvement of the game

Using the equilibrium derived above, every player receives zero income. Consider the case when players behave equally. Every player chooses $k$

as the strategy; let us find the maximum possible profit for a player. We need to find optimal points for function $k \prod_{i=1,i\neq j}^{N}(1-\frac{k}{T})$. The derivative for this function is equal to $(1-\frac{k}{T})^{N-2}(1-\frac{Nk}{T})$. The optimal point is $k=\frac{T}{N}$, and the income (if all players choose that as an optimal point) is $\frac{T}{n}(1-\frac{1}{n})^{n-1} \approx \frac{T}{n}e^{-1}$. That means that at most $\frac{T}{e}$ of the channel is divided equally (utilization $e^{-1}$ of the channel is a well-known theoretical limit for shared channels). Now, to get that optimal behavior as a Nash equilibrium for all $N$ players we need to change the income function to the following form $H_j(k_1,k_2,\dots,k_N) = k_j \prod_{i=1,i\neq j}^{N}(1-\frac{k_i}{T})1\{k_j \leq \frac{T}{N}\}$. It means that we give nothing to a player who tries to use more than $\frac{T}{N}$ of the channel. Unfortunately, this is hard to implement in practice. A known way is to add an arbiter station that jams the channel if some player uses more than it should. In that case, the income function will get the following form

$$H_j(k1,\dots,k_N) = \begin{cases} k_j \displaystyle\prod_{i=1,i\neq j}^{N}(1-\frac{k_i}{T}) & k_j < \dfrac{T}{N}, \\ \dfrac{T}{N} \displaystyle\prod_{i=1,i\neq j}^{N}(1-\frac{k_i}{T}) & k_j \geq \dfrac{T}{N}. \end{cases}$$

Unfortunately this equation does not restrict $(\frac{T}{N},\dots,\frac{T}{N})$ to be the only Nash equilibrium. A player $i$ can choose any value between $\frac{T}{N}$ and $T$.

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# Zero-sum game in the spam fighting problem

V. Mazalov, A. Lukyanenko, I. Falco, A. Gurtov

We consider the stream of tasks in Poisson manner with intensivity $\lambda$ in some information system. The service time is fixed and equal $\tau$. The priority for the service here is an offer which waits the maximal time.

There is a spumer who produces the sequence of the tasks with time interval $\theta$ and attackes the system. It yields that the waiting time of the users increases. The objective of the system is to minimize the expected delay of the users and the objective of the spumer is opposite. We find the expected delay $H(\tau, \theta)$ which depends on the parameters $\tau$ and $\theta$ and construct the equilibrium in this game.

# Two-sided best-choice game [1]

Vladimir. V. Mazalov, Anna. A. Falko

Institute of Applied Mathematical Research,
Karelian Research Centre RAS, Petrozavodsk, Russia

E-mail: vmazalov@krc.karelia.ru, afalco@krc.karelia.ru

The following two-sided best-choice game is considered. There is system with two groups of objects: computers and users. Every user has a computational task. Computers can be ordered by power and tasks can be ordered by computational complexity. Each user wants the most powerful computer to solve his task. Each computer tries to get the most computational complex task. At every stage each task randomly assigns to the computer. If the computer is satisfied by the task's complexity and the user is satisfied by the computer's power the computer solving the task. Otherwise they try to find the suitable pair at the next stage. This problem belongs to the class os two-sided best-choice games that appears in different areas of biology, sociology, market models, etc. ([1],[2])

We present the multistage game with $n+1$ stages in which objects (players) from different groups randomly meet each other at each stage. Denote $x$ the quality of computer (power) and $y$ the quality of task (computational complexity). The initial distributions of qualities are both uniform on [0, 1]. If they accept each other, they create a pair and leave the game. The aim of each player is to maximize the quality of selected object. At the last stage $n+1$ the objects who don't create the pair receive zero.

In the paper we analyze the optimal strategies in the two-sided best-choice problem and derive the explicit formulas for optimal thresholds.

All players from each group use the same strategy with thresholds $z_1, z_2, ..., z_n$. The distribution of players by quality is changing from stage

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to stage. At the beginning the distribution is uniform. The number of the players in the each group is equal to $N_0 = 1$. After the $i$-th stage the number of the players in the each group is equal to

$$N_i = 2z_i - \frac{z_i^2}{N_{i-1}}, \ i = 1, ..., n. \tag{1}$$

The distribution of players by quality after the $i$-th stage has the density of the following form:

$$f_i(x) = \begin{cases} \frac{1}{N_i}, \ 0 \le x < z_i, \\ \prod_{j=k}^{i-1} \frac{z_{j+1}}{N_j} \frac{1}{N_i}, \ z_{k+1} \le x < z_k, \ k = i-1, ..., 1, \\ \prod_{j=0}^{i-1} \frac{z_{j+1}}{N_j} \frac{1}{N_i}, \ z_1 \le x \le 1, \end{cases}$$

where $i = 1, ..., n$.

**Theorem 1.**

*Nash equilibrium in the $(n+1)$-stage two-sided best-choice game is determined by the sequence of thresholds $z_i$, $i = 1, ..., n$, which satisfy the recurrence relation*

$$z_i = a_i z_{i-1} \, i = 2, ..., n,$$

$$z_1 = \frac{1}{a_1}\left(1 - \sqrt{1 - a_1^2}\right),$$

*where coefficients $a_i$ satisfy the equations*

$$a_i = \frac{2}{3 - a_{i+1}^2}, \ i = 1, ..., n-1, \tag{2}$$

*and* $a_n = 2/3$.

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# Optimization and Game Theoretic Modeling of the Real Estate Development

G. A. Ougolnitsky

Southern Federal University, Rostov-on-Don, Russia

A system of the real estate development optimization and game theoretic models is described in this paper.

A basic role in the proposed system is played by aggregate models of a real estate development company. They are static optimization models aimed at the definition of optimal prices with constraints on the solvent demand.

A natural generalization of the basic model is possible in two directions: “horizontally” and “vertically”. First, an interaction of real estate development companies as equal economic agents may be considered. In turn, two model approaches are possible in this case. If we consider competitive relations of development companies without formation of coalitions then non-cooperative games of $n$ players in normal form arise. If a cooperation is admissible (common resources, mergers and acquisitions of development companies) then we get cooperative games.

Second, development companies have economic relations with organizations of other types. These relations are hierarchical as a rule, and a development company can be both a Leader (in relations with its suppliers) and a Follower (in relations with its investors, credit institutions, administration agencies). Respectively, hierarchical game theoretic models arise. An aggregate optimization model of a real estate development company has a form

$$u = \sum_{j=1}^{N} \left[\alpha_j\left(p_j\right) p_j - c_j\right] S_j - C \rightarrow \max \quad (1)$$

$$\sum_{j=1}^{N} \alpha_j\left(p_j\right) S_j \leq S^{\max}, \quad 0 \leq p_j \leq p_j^{\max}, \quad j = 1, \ldots, N \quad (2)$$

where $j$ is an index of a real estate development project;
$N$ is a number of projects realized by the company in the current year;
$u$ is an annual profit of the company;
$S_j$ is an annual volume of construction works in the $j$-th project ($\mathrm{m}^2$);
$c_j$ is a cost price in the $j$-th project;
$p_j$ is a sale (rent) price of $1\,\mathrm{m}^2$ in the $j$-th project;
$\alpha_j\left(p_j\right)$ is a share of the sold (rented) $\mathrm{m}^2$ in the total amount $S_j$;
$C$ are constant expenditures of the company;
$S^{\max}$ is the maximal solvent demand of the company target consumer group ($\mathrm{m}^2$);
$p_j^{\max}$ is maximal possible ("real") sale/rent price of $1\,\mathrm{m}^2$ in the $j$-th project.

Without loss of generality it is more convenient to consider the model (1)–(2) for one project, i. e.

$$u = \left[\alpha\left(p\right) p - c\right] S - C \rightarrow \max \quad (3)$$

$$\alpha\left(p\right) S \leq S^{\max}, \quad 0 \leq p \leq p^{\max} \quad (4)$$

where all variables relate to the one project.

The key role in the model (3)–(4) belongs to the variable $\alpha\left(p\right)$ which describes a dependence of a share of the sold (rented) $\mathrm{m}^2$ on the sale (rent) price. A parametrization of the function $\alpha\left(p\right)$ is based on the following assumptions:

- $\alpha\left(p\right)$ is a decreasing price function;
- let be $\alpha_{\min} \leq \alpha\left(p\right) \leq \alpha_{\max}$, then $\alpha\left(0\right) = \alpha_{\max}$, $\alpha\left(p^{\max}\right) = \alpha_{\min}$.

Two classes of functions $\alpha\left(p\right)$ were chosen for the structural identification:

- a linear function $\alpha^{(1)}(p) = a\,p + b$ $(a < 0)$;
- an exponential function $\alpha^{(2)}(p) = a\,\exp(-b\,p)$ $(a > 0,\ b > 0)$.

Using the assumptions made above we get:

$$\alpha^{(1)}(p) = p\,(\alpha_{\min} - \alpha_{\max})\,/p^{\max} + \alpha_{\max}; \quad (5)$$

$$\alpha^{(2)}(p) = \alpha_{\max}\,\exp\left(-\left(p/p^{\max}\right)[\ln\alpha_{\min} - \ln\alpha_{\max}]\right). \quad (6)$$

Solving the optimization problem (3)–(4) by Lagrange method we get for the parameterizations (5) and (6) respectively:

$$p^{(1)} = \begin{cases} \frac{p^{\max}\,(S^{\max} - \alpha_{\max}\,S)}{S\,(\alpha_{\min} - \alpha_{\max})}, & S^{\max} \le \alpha_{\max}\,S \\ p^{\max}, & S^{\max} > \alpha_{\max}\,S \end{cases} \quad (7)$$

$$p^{(2)} = \begin{cases} \frac{p^{\max}\,(\ln S^{\max} - \ln\alpha_{\max}\,S)}{S\,(\ln\alpha_{\min} - \ln\alpha_{\max})}, & S^{\max} \le \alpha_{\max}\,S \\ p^{\max}, & S^{\max} > \alpha_{\max}\,S \end{cases} \quad (8)$$

Thus, for both parameterizations the optimal solution depends on whether the inequality $S^{\max} \le \alpha_{\max}\,S$ is true. If the solvent demand is less than the supply value then the optimal price is calculated as a certain function of the model parameters. Otherwise, the company may declare an arbitrary big price restricted by common sense only.

Let on a territory there are $n$ real estate development companies designated by the index $i = 1, \ldots, n$. Then a competitive interaction of the companies is described as a non-cooperative $n$-players game in normal form

$$G = \langle\{1, \ldots, n\}, \{X_1, \ldots, X_n\}, \{u_1, \ldots, u_n\}\rangle \quad (9)$$

where payoff functions $u_i$ are given by the formula (1), and sets of admissible strategies $X_i$ are given by the constraints of a type (2). During the investigation of the game theoretic model (9) the following assumptions were studied:

1. $\alpha_i = \alpha_i(p),\ 0 \le p_i \le p_i^{\max},\ i = 1, \ldots, n,$
   where $p_i^{\max}$ is a maximal admissible sale/rent price fixed by the $i$-th company for the common sense considerations independently from others;

2. $\alpha_i = \alpha_i \left(p_i^{\text{rel}}\right), \ p_i^{\text{rel}} = p_i / p^{\max}, \ p^{\max} = \max\{p_1, \ldots, p_n\};$

3. $X_i$ is defined by constraints $\alpha_i S_i \le S_i^{\max}$ for each company $i = 1, \ldots, n$ independently;

4. $X_i$ is defined by common constraints $\sum \alpha_i S_i \le S^{\max}$ for the whole solvent demand on the territory.

In all four cases of possible combinations of the values $\alpha_i$ and $X_i$ a qualitative character of the optimal solutions (7) and (8) does not change.

As the solutions (7) and (8) are dominant strategies of the player $i$ then vectors

$$p^{(1)} = \left(p_1^{(1)}, \ldots, p_n^{(1)}\right), \quad p^{(2)} = \left(p_1^{(2)}, \ldots, p_n^{(2)}\right) \tag{10}$$

could be treated as equilibriums in dominant strategies in the game (9). But it is necessary to notice that the players' behavior is completely isolative only in the case $\alpha_i = \alpha_i(p_i), \ \alpha_i S_i \le S_i^{\max}$. In other three cases to find a dominant strategy each player must know the values of parameters of other players. That's why the solutions (10) are better to consider as Nash equilibriums which allow an informational exchange between players.

Now let on a territory there are $n$ real estate development companies $i = 1, \ldots, n$ which can exchange information, join resources and realize common projects. Denote $A_i$ an amount of own resources of the $i$-th development company.

Then we can formalize a cooperative interaction of development companies as a weighted majority game $(A^{\min}; A_1, \ldots, A_n)$, i.e. the characteristic function is

$$v(S) = \begin{cases} 1, & \sum\limits_{i \in S} A_i \ge A^{\min}, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Thus, a coalition is winning if and only if a summary amount of own resources of its members is not less than $A^{\min}$. The threshold value $A^{\min}$ can be treated as, for example, a necessary deposit for a tender or credit.

The following special cases of the game (11) can be selected:

1. a dictator game $\exists\, i \in \{1, \ldots, n\} : \; A_i \geq A^{\min},\; \forall j \neq i \; A_j < A^{\min}$.

   In this case the game is unessential, $v(S) = 1 \Leftrightarrow i \in S$, the only imputation $(0, \ldots, 0, 1, 0, \ldots, 0)$ $(x_i = 1)$ exists which forms $C$-core, is the only Neumann-Morgenstern solution and the Shapley value;

2. a symmetrical game of the $k$-th order

$$v(S) = \begin{cases} 1, & s \geq k, \\ 0, & \text{otherwise.} \end{cases} \qquad s = |S|, \quad 1 \leq k \leq n.$$

   In this case the $C$-core is empty, the Shapley value has a form $(1/n, \ldots, 1/n)$, an example of the Neumann-Morgenstern solution is given by a discriminative solution
   $\{(x_{i\,1}, \ldots, x_{i\,k}, 0, \ldots, 0) : \; x_{i\,1} \geq 0, \ldots, x_{i\,k} \geq 0; \; x_{i\,1} + \ldots + x_{i\,k} = 1\}$.

An interaction of real estate development companies with a bank (let's suppose for simplicity that there is only one bank on the territory) is described by the following rules.

Stage 1: preparation of the credit applications by development companies.

This stage includes for each company $i = 1, \ldots, n$:

- forming of the concepts for projects $j = 1, \ldots, n_i$;
- working out of schedules of the project works, construction works, financing for each project;
- evaluating of own resources and cost price per $1\,\mathrm{m}^2$ for each project;
- exposing the credit needs and application to the bank with the request

$$K_i^0 = \sum_{j=1}^{n_i} K_{ij}^0.$$

Stage 2: decision making by the bank. At this stage the bank:

- analyzes the requests $K_1^0, \ldots, K_n^0$;
- evaluates of the credit risks $r_i$ for each request;

- defines a rate of interest $s_i = s_i\,(r_i)$;
- makes a decision about credits $K_1, \ldots, K_n$ and rates of interest $s_1, \ldots, s_n$;
- informs development companies about the decision.

Stage 3: decision making by a development company.
At this stage each development company $i = 1, \ldots, n$:

- specifies real amounts of the construction works and respective schedules based on given credit resources $K_i$ and rate of interest $s_i$;
- calculates the optimal price for development objects by solving the optimization problem (3)–(4).

The following assumptions are made to build a model of decision making by the bank:

- the credit risk is defined by the formula

$$r_i = K_i/A_i, \quad i = 1, \ldots, n, \tag{12}$$

  where $A_i$ are own resources of the $i$-th company, $K_i$ are credit resources assigned by the bank. Then a condition of credit apportionment is an inequality $r_i \leq r^{\max}$, where $r^{\max}$ is a banking normative of admissible risk;

- the interest rate is an increasing linear function of the risk: $s_i = a\,r_i + b = a\,K_i/A_i + b = a_i\,K_i + b,\ i = 1, \ldots, n$. Let's consider that

$$0 < s_{\min} \leq s_i \leq s_{\max} < 1,\ r_{\min} \leq r_i \leq r_{\max},\ s\,(r_{\min}) = s_{\min},$$

$$s\,(r_{\max}) = s_{\max}.$$

Then we get

$$a_i = \frac{s_{\max} - s_{\min}}{A_i\ (r_{\max} - r_{\min})}, \quad b = \frac{s_{\min}\,r_{\max} - s_{\max}\,r_{\min}}{r_{\max} - r_{\min}}, \quad i = 1, \ldots, n. \tag{13}$$

Considering the assumptions made above the model of decision making by the bank at the stage 2 is an optimization problem

$$u_0 = \sum_{i=1}^{n} s_i\, K_i = \sum_{i=1}^{n} (a_i\, K_i + b)\, K_i \to \max \tag{14}$$

$$\sum_{i=1}^{n} K_i \le K, \quad 0 \le K_i \le L_i, \quad j = 1, \ldots, n \tag{15}$$

where $K$ is a whole capital of the bank in the current year, $L_i = \min\left\{K_i^0, A_i\, r^{\max}\right\}$. Solving the problem (14)–(15) by Lagrange method we find the optimal values

$$K_i^* = \min\left\{L_i, M_i\right\}, \qquad M_i = K / \left(a_i \sum a_i^{-1}\right); \tag{16}$$

$$s_i^* = \frac{(s_{\max} - s_{\min})\, K_i^* + A_i (s_{\min}\, r_{\max} - s_{\max}\, r_{\min})}{A_i (r_{\max} - r_{\min})}, \quad i = 1, \ldots, n. \tag{17}$$

The model of decision making by an $i$-th development company at the stage 3 has a form (3)–(4) with an additional constraint

$$c_i\, S_i \le A_i - C_i + (1 - s_i^*)\, K_i^*, \tag{18}$$

from what we get a final value of the optimal construction works amount

$$S_i^* = \left[A_i - C_i + (1 - s_i^*)\, K_i^*\right] / c_i, \tag{19}$$

which has to be substituted instead of $S$ in the formulas (7)–(8) to calculate the optimal prices.

Let's consider a case $n = 1$. The rules described above define a hierarchical game "Bank–Developer" in the form:

$$u_0\,(K_1) = a_1\, K_1^2 + b\, K_1 \to \max \tag{20}$$

$$0 \le K_1 \le \min\left\{K, K_1^0, A_1\, r^{\max}\right\} \tag{21}$$

$$u_1\,(K_1, p_1) = \left[\alpha_1\,(p_1)\; p_1 - c_1\right] \left[A_1 - C_1 + (1 - s_1)\; K_1\right] / c_1 \to \max \tag{22}$$

$$0 \le \alpha_1\,(p_1)\, \left[A_1 - C_1 + (1 - s_1)\; K_1\right] / c_1 \le S_1^{\max},\; 0 \le p_1 \le p_1^{\max}. \tag{23}$$

The outcome $(K_1^*, p_1^*)$, where $K_1^*$ is calculated by the formula (16), and $p_1^*$ by one of the formulas (7) or (8) after substitution of the values $s_i^*$ and $S_i^*$ by formulas (17) and (19) respectively, is a formal Stackelberg equilibrium in the game (20)–(23). But this game is degenerate because Bank's payoff function does not depend on $p_1$ and the Bank chooses the solution $K_1^*$ by solving the optimization problem (14)–(15) and is not interested in Developer's optimal reaction.

# Multi-Player Network Game[1]

Anna N. Rettieva

Institute of Applied Mathematical Research,
Karelian Research Centre RAS, Petrozavodsk, Russia

**Abstract**

In this work we present some game-theoretic models based on player rating. We used goodness variables to encourage good behavior of network players. We derive Nash equilibria for finite planning horizon in all our models. The numerical modelling and the results comparison are given.

## 1. Model with goodness function

We consider game-theoretic model based on player rating. In [2] it was introduced a goodness function which represents player's behavior history as a value from $(0,1)$. This function presents a rating of a player and central server takes it into consideration in the serving process.

In our model we have two players. The system starts executing at time moment 0 and stops at $T$. Each player interacts only with central server and demands some service from it. Every player has a goodness variable $x_i(t) \in (0,1)$, $x_i(0) = x_0^i$. The central server gives a player $i$ a service value proportional to his goodness value. A player can get at most twice more what the server suggests. Hence the players' controls $u_1(t), u_2(t) \in (0,2]$ correspond to the factor of what server suggests to take.

Goodness variables depend on players' behavior and change according to the following rule:

$$\begin{cases} x_1'(t) = x_1(t)(1-x_1(t))\Big(\dfrac{1}{2} - \dfrac{u_1(t)}{u_1(t)+u_2(t)}\Big)\,,\ x_1(0) = x_0^1\,, \\ x_2'(t) = x_2(t)(1-x_2(t))\Big(\dfrac{1}{2} - \dfrac{u_2(t)}{u_1(t)+u_2(t)}\Big)\,,\ x_2(0) = x_0^2\,, \end{cases}$$

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where $0 \le u_1,\, u_2 \le 2$ – players' demands, $0 < x_1, x_2 < 1$ – players' goodness variables.

We can notice that player's $i$ goodness value has a strong connection with player's $j$ control. The player's $i$ rating decreases at time moment $t$ only if $u_j(t) < u_i(t)$, i.e. player's $j$ behavior at this time moment is better for the central server.

Players' net revenues over finite time horizon are:

$$J_1 = \int_0^T x_1(t)u_1(t)\,dt\,, \;\; J_2 = \int_0^T x_2(t)u_2(t)\,dt\,.$$

Players act non-cooperatively and wish to maximize their payoffs. We find Nash equilibrium using Pontryagin maximum principle.

**2. Model with relative goodness function**

We change the model in the sense that now players determine the goodness values for each other. Let $x_{ij} \in (0,1)$ be the relative goodness function, i.e. player $i$ defines a rating of player $j$. Of course $x_{ii} = 1$, $i = 1, 2$, because player $i$ thinks good about himself. Again the central server gives a player $i$ a service value proportional to his relative goodness value.

Players' payoffs are:

$$J_1 = \int_0^T u_1(t)\frac{x_{21}(t)}{x_{21}(t)+1}\,dt\,, \;\; J_2 = \int_0^T u_2(t)\frac{x_{12}(t)}{x_{12}(t)+1}\,dt\,,$$

where $0 \le u_1,\, u_2 \le 2$ – players' demands, $0 < x_{12}, x_{21} < 1$ – players' relative goodness variables.

Relative goodness variables change according to the following rule:

$$\begin{cases} x'_{12}(t) = x_{12}(t)(1-x_{12}(t))\Big(\dfrac{1}{2} - \dfrac{u_2(t)}{u_1(t)+u_2(t)}\Big)\,, \; x_{12}(0) = x_0^2\,, \\ x'_{21}(t) = x_{21}(t)(1-x_{21}(t))\Big(\dfrac{1}{2} - \dfrac{u_1(t)}{u_1(t)+u_2(t)}\Big)\,, \; x_{21}(0) = x_0^1\,. \end{cases}$$

For this model we also determine Nash equilibrium. Also we extend this model for the game with three players.

**3. Combined model**

The last model we consider here is the combination of first two. Now we have two goodness variables for each player: $x_i \in (0,1)$ – central server's rating of player $i$ and $x_{ji} \in (0,1)$ – rating of player $i$ which gives him player $j$, $i,j = 1,2$.

Here players' net revenues are:

$$J_1 = \int_0^T u_1(t)\Big(x_1(t) + \frac{x_{21}(t)}{x_{21}(t)+1}\Big)\,dt\,,$$

$$J_2 = \int_0^T u_2(t)\Big(x_2(t) + \frac{x_{12}(t)}{x_{12}(t)+1}\Big)\,dt\,,$$

where $0 \le u_1,\, u_2 \le 2$ – players' demands, $0 < x_1, x_2 < 1$ – players' goodness variables, $0 < x_{12}, x_{21} < 1$ – players' relative goodness variables.

Goodness variables change according to

$$\begin{cases} x_1'(t) = x_1(t)(1-x_1(t))(1-u_1(t))\,,\; x_1(0) = x_0^1\,, \\ x_2'(t) = x_2(t)(1-x_2(t))(1-u_2(t))\,,\; x_2(0) = x_0^2\,, \\ x_{12}'(t) = x_{12}(t)(1-x_{12}(t))\Big(\dfrac{1}{2} - \dfrac{u_2(t)}{u_1(t)+u_2(t)}\Big)\,,\; x_{12}(0) = x_0^{12}\,, \\ x_{21}'(t) = x_{21}(t)(1-x_{21}(t))\Big(\dfrac{1}{2} - \dfrac{u_1(t)}{u_1(t)+u_2(t)}\Big)\,,\; x_{21}(0) = x_0^{21}\,. \end{cases}$$

Numerical modelling was carried out for all presented models and we compare player's controls and payoffs.

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# Various models of on line disorders detections

Wojciech Sarnowski and Krzysztof Szajowski

Institute of Mathematics and Computer Science, Wroclaw University of Technology, Poland

The paper deals with an on-line detection disorder problem (see Shiryaev [4]) under probability maximizing of abrupt changes localization approach to the sequences which are not necessarily i.i.d. before and after the disruption moment. Some problems with such generalization have been touched by Moustakides [2]. The considerations are inspired by the problem regarding how can we protect ourselves against a second fault in a technological system after the occurrence of an initial fault (see Szajowski[5]). At two random moments $\zeta$, $\eta$, where $\zeta < \eta$, the distribution of observed sequence changes. It is known before $\zeta$ and after $\eta$. Between these instants is unknown to the statistician and chosen randomly by "nature" from a set of distributions (see e.g. Bojdecki et al. [1], Sarnowski & Szajowski [3]). The stopping rule which stops between disorder moments $\zeta$ and $\eta$ with maximal probability is identified.

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# Organization of the state inspections and suppression of corruption[1]

Alexander Vasin, Anton Urazov

Lomonosov Moscow State University, Moscow, Russia

The state inspections play an important role in the modern economy. There are two main directions of their activity. The first one is collection of payments to the state budget. The tax inspections and the customs control the payment's values and check exemptions from payments for different economic agents. The agency should prevent tax or customs evasion but not interfere with the agents eligible for exemption from the payment. The second direction is concerned with prevention of the law infringement. Police, sanitary, firework inspection and others deal with this task. The efficiency of an inspection should be measured by the social welfare increase proceeding from its activity.

For many countries in transition, in particular for Russia, corruption is the most important problem in inspections' organization. Bribery is one form of corruption that is the most difficult to reveal. There exists a wide literature that discusses problems of optimal inspection organization (in particular for tax inspection) and the problem of corruption. The first type of models (see Srinivasan (1973)) studies the interaction between the tax authority and a group of taxpayers, whose income is random, without taking into account the possibility of corruption. It is assumed that at the end of the accounting period each taxpayer declares his/her income to the tax inspectors. The reported income is taxed according to the given tax rates. However, a taxpayer may try to hide some part of income by under-reporting. If the taxpayer is audited, the inspector will inevitably uncover the true level of income. The detected tax evader is fined and made to pay the evaded tax. Further, it is assumed that auditing is costly and that

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the central authority is interested in maximizing net tax revenue (i.e. the sum of taxes and penalties minus expenditures on audits) given the tax rates, fines and the costs of auditing. In the case of a homogeneous group of taxpayers, the only taxpayer-specific information available to the tax authority is the declared incomes. Thus, the authority must determine the probability of audit, using these declarations. The purpose of this model is to find the optimal auditing rule given the tax rates and income distribution.

Chander, Wilde (1992) and Vasin, Panova (2000) extend the previous model by taking corruption into account. The model assumes that a tax inspector, which has discovered an instance of tax evasion, may bargain with the detected evader over the size of a bribe given in exchange for not revealing the evasion. In order to prevent this kind of corruption, the authority chooses to review some of the inspectors' audits and fires those inspectors who have not reported tax evasion. Thus, the authority's problem is to choose the frequencies of both levels of audit - the audit of taxpayers by inspectors and the review of audits from the center as well as inspectors salary. There are two variants of the optimal strategy depending on parameters of the model:

1. If the ratio of the audit cost to the cost of reviewing is above some threshold then the optimal strategy includes threshold probabilities of auditing and reviewing that make corruption and tax evasion unprofitable.

2. If the ratio is below this threshold level then it is optimal to cancel reviewing and increase the auditing probability to such value that tax evasion turns out to be unprofitable in spite of the possibility for bribing.

However, realization of these variants meets the following difficulties:

1. The first variant assumes that there is a possibility to hire sufficient number of honest collaborators for reviewing, but actually the center typically has very few reliable collaborators and their time is a very expensive resource. Thus, this variant may be impossible or inefficient.

2. As to the second variant, the lack of control creates incentives for cooperation among inspectors in order to reduce the actual auditing probability to such value that maximizes the total amount of bribes.

An alternative approach is to form a controlling hierarchy that suppresses corruption at all levels. Consider a country where a benevolent leader aims to organize an efficient tax collection. There are $N$ firms, each gets high or low income with probabilities $h$ and $1-h$ respectively. The additional tax from the high income is $T$ and the penalty for evasion is $F$. For the inspection, the leader can use a small number $M$ of reliable collaborators and also employ any number of rational inspectors who maximize their expected incomes with account of possible salaries, bribes and penalties. Salary $s_M$ (per one audit or review) permits to employ a sufficient number of such inspectors, and $\tilde{c}$ is the cost of one audit by a reliable collaborator. Consider a strategy of the tax inspection organization. It includes probability $p_0$ of primary audit for any low-income declaration. In order to prevent bribing of a primary auditor, any report confirming low income is under reviewing (first-level audit) with probability $p_1$. And so on, any $i$-level audit confirming the low income is under reviewing ($i+1$-level audit) with probability $p_{i+1}$ until the upper level $k$ where honest collaborators work. A salary of an i-level inspector is $s_i \geq s_M$. Each revealed inspector which has not reported tax evasion is fired and gets after that alternative salary $s_{alt}$. This value is uncertain: we assume that $s_{alt} \in (s_M - \Delta, s_M)$. Thus, a government strategy includes the number $k+1$ of audit levels, auditing probabilities $p_0, \ldots, p_k$ and salaries $s_0, \ldots, s_k$ at each level.

A formal problem is to find the optimal strategy that provides honest behavior of all agents and maximizes net tax revenue under this condition. Note that, for risk-neutral inspector, firing as equivalent to monetary fine $\tilde{F} = (s - s_{alt})\alpha$, $\alpha = \delta/(1-\delta)$, where $\delta$ is a discount coefficient. Let $d_i = s_i - s_M$ denote the increment of the salary at level $i$ above the maximum alternative salary.

**Proposition 1** *Assume that auditors at level i check honestly. Then mutually beneficial collusion between $i-2$-level inspector and his auditor is impossible if and only if* $p_i \geq \dfrac{d_{i-2}+\Delta}{d_{i-2}+d_{i-1}+\Delta}$ *for* $i = 2, \ldots, k$. *Tax evasion is unprofitable if and only if* $p_0 \geq T/F$ *and collusion between taxpayer* *and his auditor is impossible if and only if* $p_1 \geq \dfrac{F}{F + d_0\alpha}$. *(*)*

**Proposition 2** *The subgame perfect equilibrium corresponding to the honest behavior in the interaction of inspectors and taxpayers exists if and only if the government strategy meets the inequalities in the previous proposition. The net tax revenue at such equilibrium is as follows:*

$$R(k, \overrightarrow{d}) = hT - p_0(1-h)\cdot$$

$$\cdot(s_M + d_0 + p_1(s_M + d_1 + p_2(\cdots + p_{k-1}(s_M + d_{k-1} + p_k\tilde{c})\dots).$$

Consider the following example. The additional tax from the high income is 10 000 and the penalty for evasion is 80 000. The number of taxpayers is 100 000, the probability to get high income is $h = 0{,}5$. Reliable collaborators get 100 000 per one check. Salary $s_M$ equals 150 and $\Delta$ equals 100, so $s_{alt} \in (50, 150)$. Each auditor can make 60 inspections or revisions per year. So his alternative salary per year lies between 3 000 and 9 000. Let a discount coefficient $\delta$ equal 0,1.

The following table shows the net tax revenue and auditing expenses for optimal salaries, probabilities and different number of auditing levels.

| Number of auditing levels | Net tax revenue | Auditing expenses | Number of employed honest collaborators |
|---|---|---|---|
| 2 | 181 940 000 | 318 060 000 | 1863 |
| 3 | 459 222 000 | 40 778 000 | 200 |
| 4 | 480 532 000 | 19 468 000 | 62 |
| 5 | 488 653 000 | 11 347 000 | 25 |
| 6 | 491 774 000 | 8 226 000 | 12 |
| 7 | 493 424 000 | 6 576 000 | 6 |
| 8 | 494 215 000 | 5 785 000 | 4 |
| 9 | 494 695 000 | 5 305 000 | 2 |
| 10 | 494 947 000 | 5 053 000 | 2 |

According to this data, the 6-level inspection organization cuts down auditing expenses 40 times with respect to the base model with 2 levels. Moreover, the necessary number of honest collaborators also decreases by 150 times. So even a small number of honest collaborators can provide an efficient tax audit.

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# Strong equilibrium in a linear-quadratic stochastic differential game

Nikolay A. Zenkevich and Andrey V. Zyatchin

Graduate School of Management
Volkhovsky per.3, Saint-Petersburg, Russia

E-mail: zenkevich@gsom.pu.ru, zyatchin@gsom.pu.ru

**Abstract**

Sufficient conditions for strong equilibrium to exist in a differential game with stochastic controllable dynamics are formulated. An example is proposed, where linear-quadratic game was solved by reducing it to the optimal control problem.

## 1 Problem statement

Consider stochastic differential game with many players $\Gamma(x_0, T-t_0)$. Initial state is $x_0$ and the duration is $T-t_0$, where $t_0$, $T$ — moments of beginning and ending of the game. Denote a set of players as $N = \{1,\ldots,i,\ldots n\}$, $n \geq 2$. Stochastic dynamic is:

$$dx(\tau) = f(\tau, x(\tau), u_1(\tau), \ldots, u_n(\tau))dt + \sigma(\tau, x(\tau), u_1(\tau), \ldots, u_n(\tau))dz(\tau), \tag{1}$$

where $x(t_0) = x_0$, $z(\tau)$ is a state of Brownian motion [1, 2, 3], $x(\tau) \in R$ is a game state variable, $u_i(\tau)$ — player's $i \in N$ control at the moment $\tau$, $u_i \in U_i \subset R$, $\prod\limits_{i\in N} U_i = U_N \subset R^n$.

Suppose that functions $f(\tau, x(\tau), u_1(\tau), \ldots, u_n(\tau))$, $\sigma(\tau, x(\tau), u_1(\tau), \ldots, u_n(\tau))$ are continuously differential on $[t_0, T] \times R \times$

$U_N$. Let the object of every player $i \in N$ is a maximization of expected value of the functional [4,5]:

$$\max_{u_i} E_{t_0}\left[\int_{t_0}^{T} g_i(\tau, x(\tau), u_1(\tau), \ldots, u_i(\tau), \ldots u_n(\tau))d\tau + q_i(x(T))\right], \quad i \in N \tag{2}$$

where $g_i(\tau, x(\tau), u_1(\tau), \ldots, u_i(\tau), \ldots u_n(\tau))$ and $q_i(x(T))$ — continuous functions.

Consider games with perfect information [5]. We will find a solution in the class of feed-back strategies. A feed-back strategy $\varphi_i(\tau, x(\tau))$ of player $i$ has following program realization: $u_i(\tau) = \varphi_i(\tau, x(\tau))$, $u_i(\tau) \in U_i$, $\tau \in [t_0, T]$. Let $S \subset N$ is an arbitrary coalition in the game $\Gamma(x_0)$. Denote strategy of coalition $S$ as $\varphi_S(\tau, x) = (\varphi_i(\tau, x))_{i\in S} \in \prod_{i\in S} U_i = U_S \subset R^s$, $\tau \in [t_0, T]$, $s = |S|$. Let $\varphi(\tau, x) = (\varphi_1(\tau, x), \ldots \varphi_n(\tau, x))$ is a situation in feed-back strategy. A payoff of the coalition $S$ is a sum of payoffs:

$$J_S(x_0, \varphi(\tau, x)) = \sum_{i\in S} J_i(x_0, \varphi(\tau, x)) =$$

$$E_{t_0}\left[\int_{t_0}^{T} g_S(\tau, x(\tau), \varphi(\tau, x))d\tau + q_S(x(T))\right],$$

where $g_S(\tau, x(\tau), \varphi(\tau, x)) = \sum_{i\in S} g_i(\tau, x(\tau), \varphi(\tau, x))$.

We use strong equilibrium optimality principle as a solution of the game $\Gamma(x_0, T - t_0)$ [6, 7].

**Definition 1** *A couple* $\{\varphi_1^*(\tau, x), \varphi_2^*(\tau, x), \ldots, \varphi_n^*(\tau, x)\}$, $\tau \in [t_0, T]$ *we will call Strong equilibrium in the game* $\Gamma(x_0, T - t_0)$, *if for any coalition* $S \subset N$, $S \neq \emptyset$ *and strategy* $\varphi_S(\tau, x) \in U_S$ *the following inequalities take place:*

$$E_{t_0}\left[\int_{t_0}^{T} g_S(\tau, x^*(\tau), \varphi_S^*(\tau, x), \varphi_{N/S}^*(\tau, x))d\tau + q_S(x^*(T))\right] \geq$$

$$\geq E_{t_0}\left[\int\limits_{t_0}^{T} g_S(\tau, x^{[S]}(\tau), u_S(\tau), \varphi^*_{N/S}(\tau, x^{[S]}))d\tau + q_S(x^{[S]}(T))\right],$$

*where*

$$\begin{aligned} dx^*(\tau) &= f(\tau, x^*(\tau), \varphi^*_S(\tau, x), \varphi^*_{N/S}(\tau, x))d\tau \\ &+ \sigma(\tau, x^*(\tau), \varphi^*_S(\tau, x), \varphi^*_{N/S}(\tau, x))dz(\tau), \end{aligned}$$

$$x^*(t_0) = x_0.$$

$$\begin{aligned} dx^{[S]}(\tau) &= f(\tau, x^{[S]}(\tau), \varphi_S(\tau, x^{[S]}), \varphi^*_{N/S}(\tau, x^{[S]}))d\tau \\ &+ \sigma(\tau, x^{[S]}(\tau), \varphi^{[S]}_S(\tau, x^{[S]}), \varphi^*_{N/S}(\tau, x^{[S]}))dz(\tau), \end{aligned}$$

$$x^{[S]}(t_0) = x_0.$$

## 2 The results

**Theorem 1** *Suppose that for any coalition $S \subset N$, $S \neq \emptyset$ there exist double continuous-differentiable functions $V^{[S]}(t,x)$ and a couple $\{\varphi^*_i(t, x(t)) \in U_i,\, i \in N\}$, satisfying the following system of Bellman-Isaaks equations:*

$$V^{[S]}_t(t, x^{[S]}) + \max_{u_S}\left\{\frac{1}{2}\sigma^2\left(t, x^{[S]}, u_S(t), \varphi^*_{N/S}(t, x^{[S]})\right)V^{[S]}_{xx}\left(t, x^{[S]}(t)\right) + \right.$$

$$+f\left(t, x^{[S]}, u_S(t), \varphi^*_{N/S}(t, x^{[S]})\right)V^{[S]}_x\left(t, x^{[S]}\right) +$$

$$\left. + g_S\left(t, x^{[S]}, u_S(t), \varphi^*_{N/S}(t, x^{[S]})\right)\right\} =$$

$$= V^{[S]}_t(t, x^*) + \frac{1}{2}\sigma^2(t, x^*, \varphi^*(t, x^*))V^{[S]}_{xx}(t, x^*) +$$

$$+f(t, x^*, \varphi^*(t, x^*))V^{[S]}_x\left(t, x^{[S]}\right) + g_S(t, x^*, \varphi^*(t, x^*)) = 0$$

*with dynamics*

$$dx^*(t) = f\left(t, x^*, \varphi^*(t, x^*)\right) dt + \sigma\left(t, x^*, \varphi^*(t, x^*)\right) dz,$$

$$x^*(t_0) = x_0, \quad u_S(t) \in U_S, \quad V^{[S]}(T, x^{[S]}) = q_S(x^{[S]}(T)).$$

*then for any initial conditions* $[t_0, x_0]$ *the couple* $\{\varphi_i^*(t, x(t)) \in U_i, \quad i \in N\}$ *constitutes strong equilibrium in (1)-(2).*

Consider an example of linear-quadratic stochastic game, where the solution was found by reducing an original statement to optimal control problem.

**Example**. Consider a differential game with dynamics

$$dx(t) = \left(ax + \sum_{i=1}^{3} b_i u_i\right) dt + \sigma x dz, \quad x(t_0) = x_0, \tag{3}$$

where $u_i \in R$; $a$, $b_i$, $\sigma$ — are known parameters, $i \in N$.

The objective function of player $i \in N = \{1, 2, 3\}$ is

$$J_i(x_0, u) =$$
$$E_{t_0}\left[\int_{t_0}^{T}\left(r_i(t) - \sum_{i=1}^{3} h_i u_i^2 + x\sum_{i=1}^{3} u_i - \frac{x^2}{4}\left[\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3}\right]\right) dt + hx(T)\right]$$

where $r_i(t)$ — continuous on $[t_0, T]$ function. A payoff of the coalition $S$ is a sum of players' $i \in S$ payoffs. Then in the game (3)-(4) there exists strong equilibrium.

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