Piecewise affine functions Sturmian sequences & Aperiodic tilings

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Outline of the talk

- Wang tiles
- Aperiodicity. An aperiodic tile set of 14 Wang tiles
- Tiles to simulate piecewise affine transformations
- Undecidability of the tiling problem
- The tiling problem on the hyperbolic plane

Wang tiles

A Wang tile is a unit square tile with colored edges. A tile set T is a finite collection of such tiles. A valid tiling is an assignment

$$\mathbb{Z}^2 \longrightarrow T$$

of tiles on infinite square lattice so that the abutting edges of adjacent tiles have the same color.

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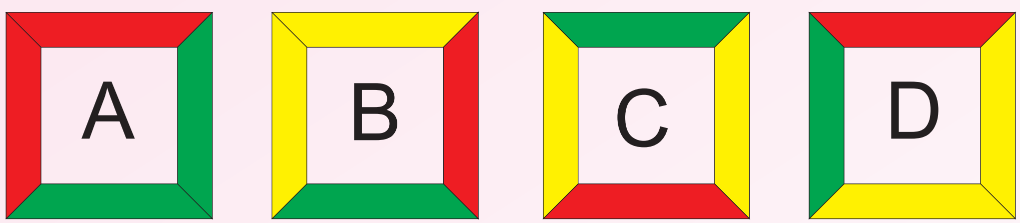
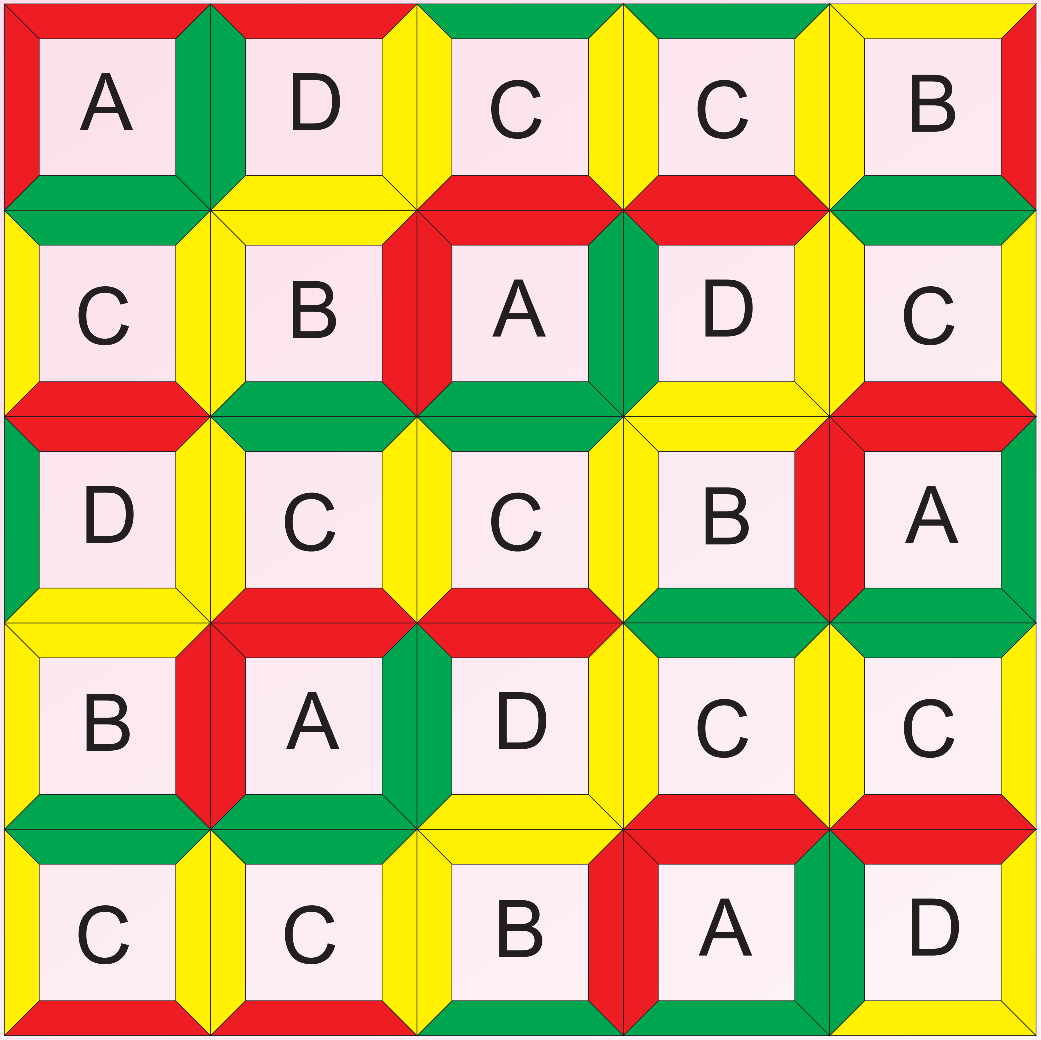
For example, consider Wang tiles



With copies of the given four tiles we can properly tile a 5×5 square. . .



. . . and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the plane.

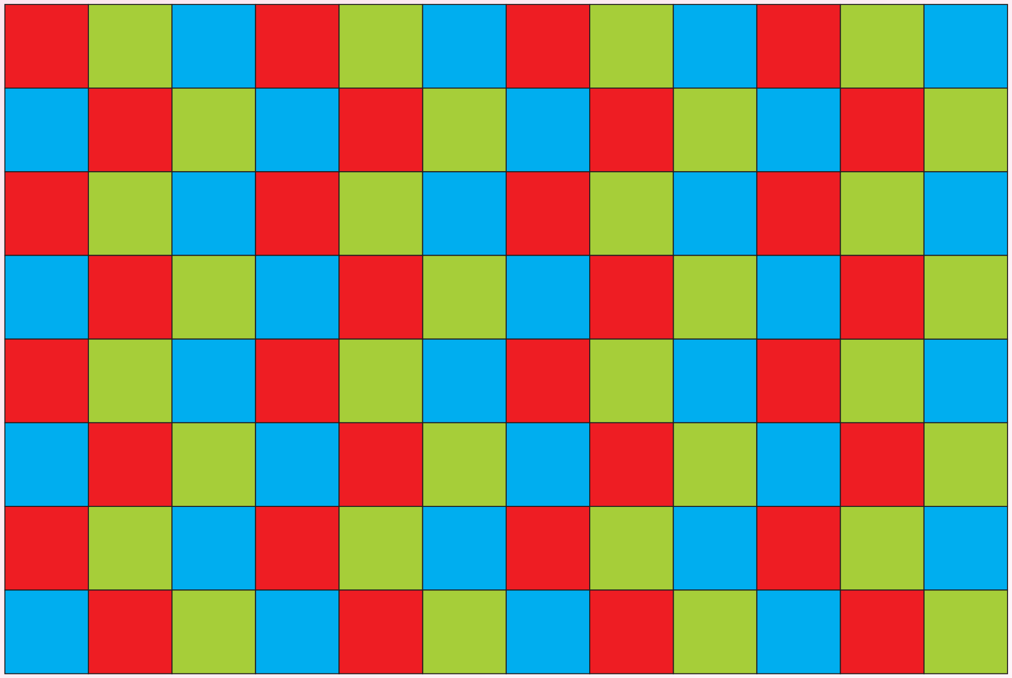
The **tiling problem** of Wang tiles is the decision problem to determine if a given finite set of Wang tiles admits a valid tiling of the plane.

Theorem (R.Berger 1966): The tiling problem of Wang tiles is undecidable.

Aperiodicity

A tiling is called **periodic** if it is invariant under some non-zero translation of the plane.

A Wang tile set that admits a periodic tiling also admits a doubly periodic tiling: a tiling with a horizontal and a vertical period:



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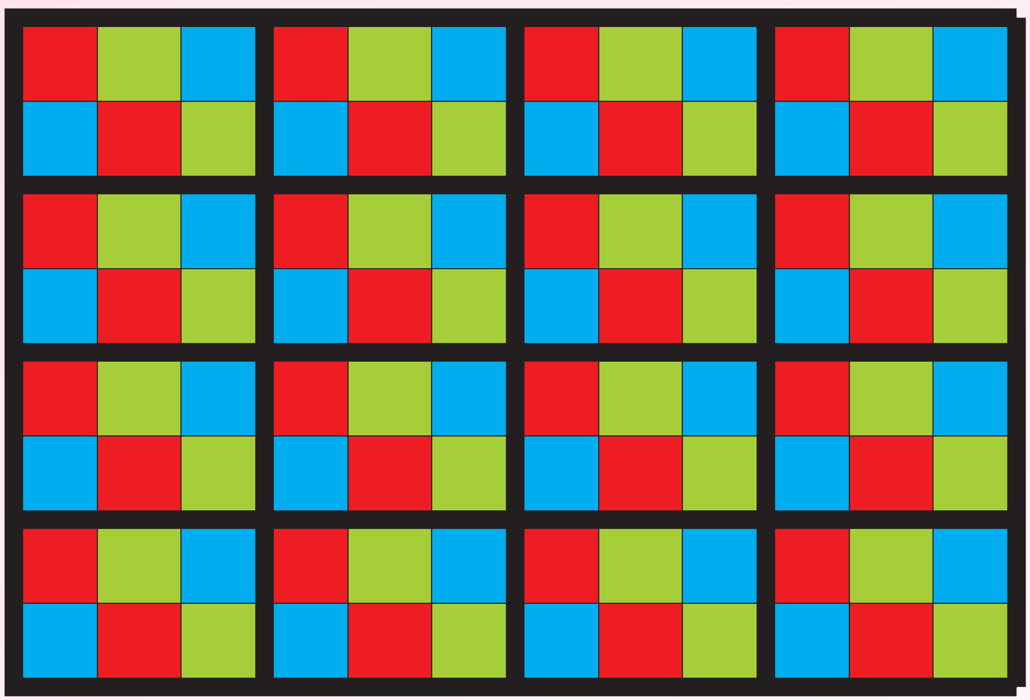

Conjecture by **H. Wang** in the 50's:

$$T \text{ admits tiling } \implies T \text{ admits periodic tiling.}$$

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There is a tile set that admits a tiling but does not admit periodic tilings.

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Such tile sets are called **aperiodic**.

Berger's aperiodic tile set contained 20,426 tiles.

Current record: 13 tiles (by K. Culik) based on the method of this talk.

Remark: If Wang's conjecture had been true then the tiling problem would be decidable:

Try all possible tilings of larger and larger rectangles until either

(a) a rectangle is found that can not be tiled (so no tiling of the plane exists), or

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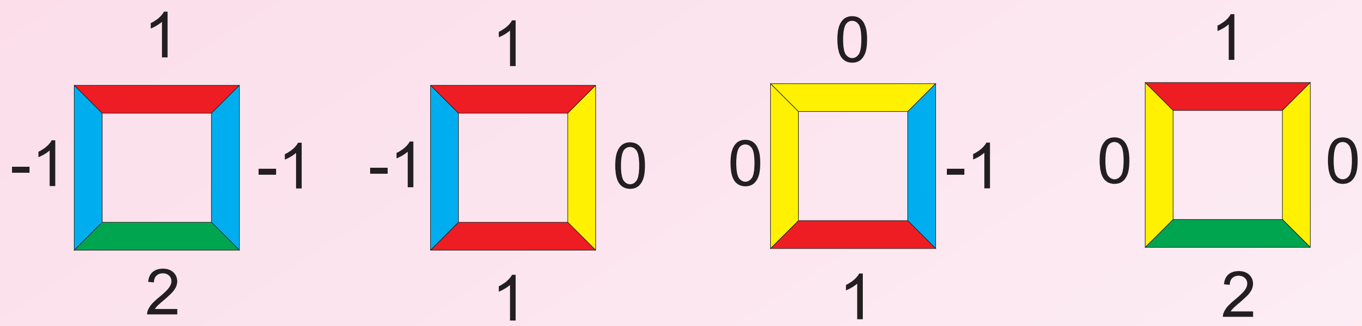
Any undecidability proof of the tiling problem must contain (explicitly or implicitly) a construction of an aperiodic tile set.

14 tile aperiodic set

The colors in our Wang tiles are real numbers, for example



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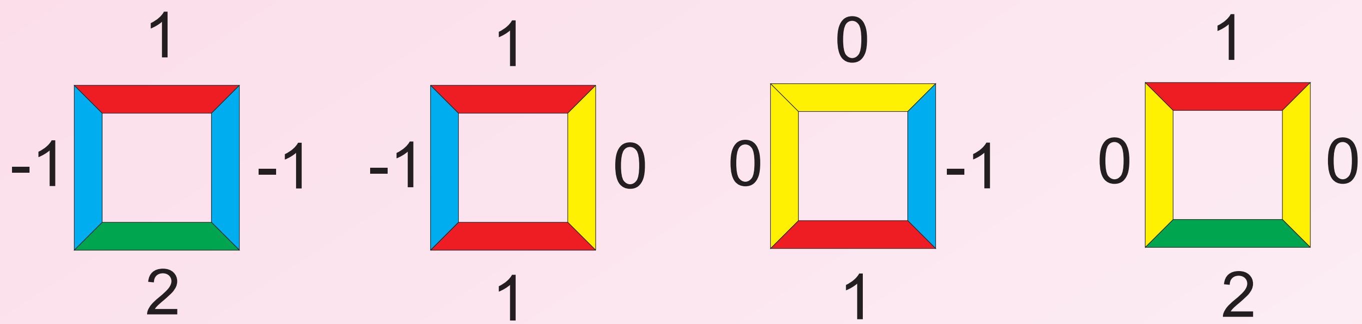
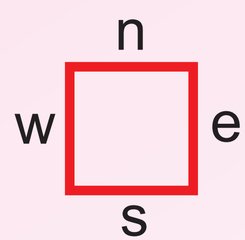


We say that tile



multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

(The **"input"** n comes from the north, and the **"carry-in"** w from the west is added to the product qn. The result is split between the **"output"** s to the south and the **"carry-out"** e to the east.)

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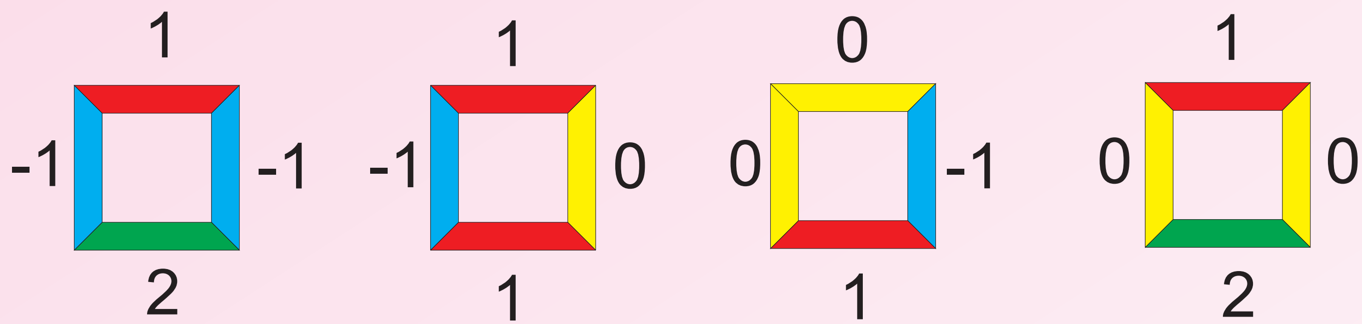

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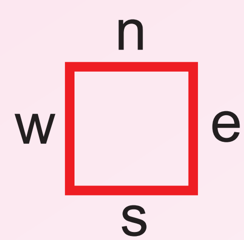
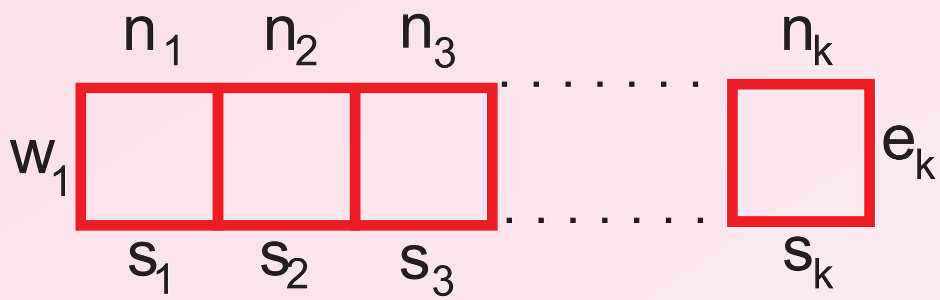


multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

The four sample tiles above all multiply by $q = 2$.

Suppose we have a correctly tiled horizontal segment where all tiles multiply by the same q.



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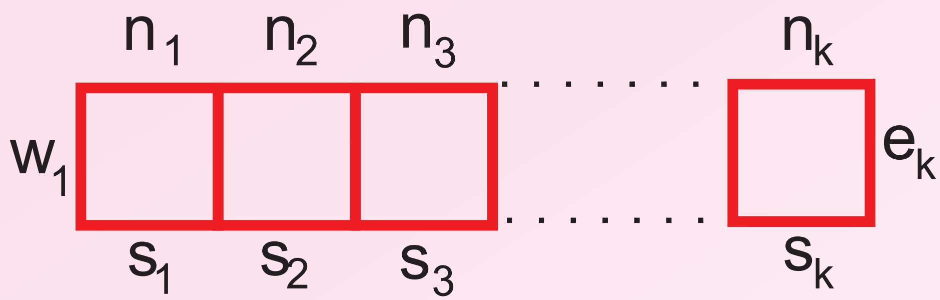

Adding up

$$\begin{aligned} qn_1 + w_1 &= s_1 + e_1 \\ qn_2 + w_2 &= s_2 + e_2 \\ &\vdots \\ qn_k + w_k &= s_k + e_k, \end{aligned}$$

taking into account that $e_i = w_{i+1}$ gives

$$\mathbf{q(n_1 + n_2 + \ldots + n_k) + w_1 = (s_1 + s_2 + \ldots + s_k) + e_k.}$$

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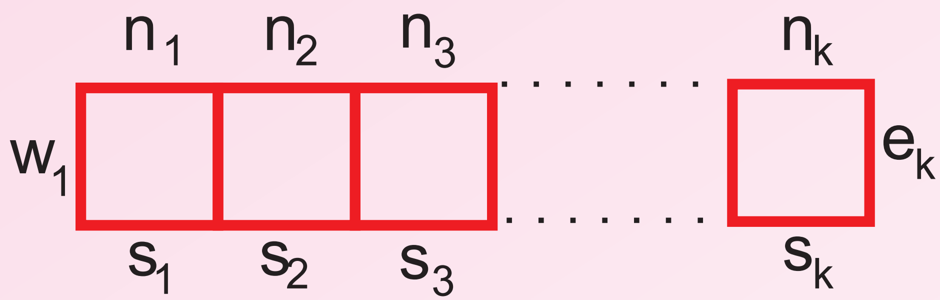
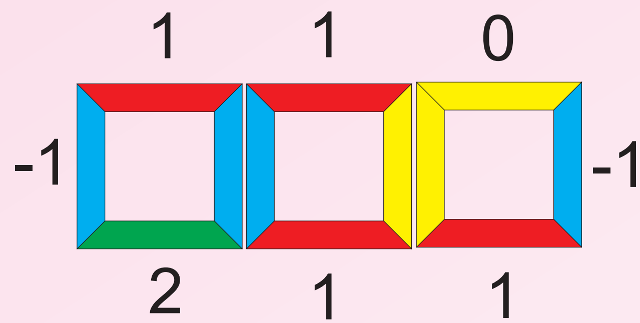


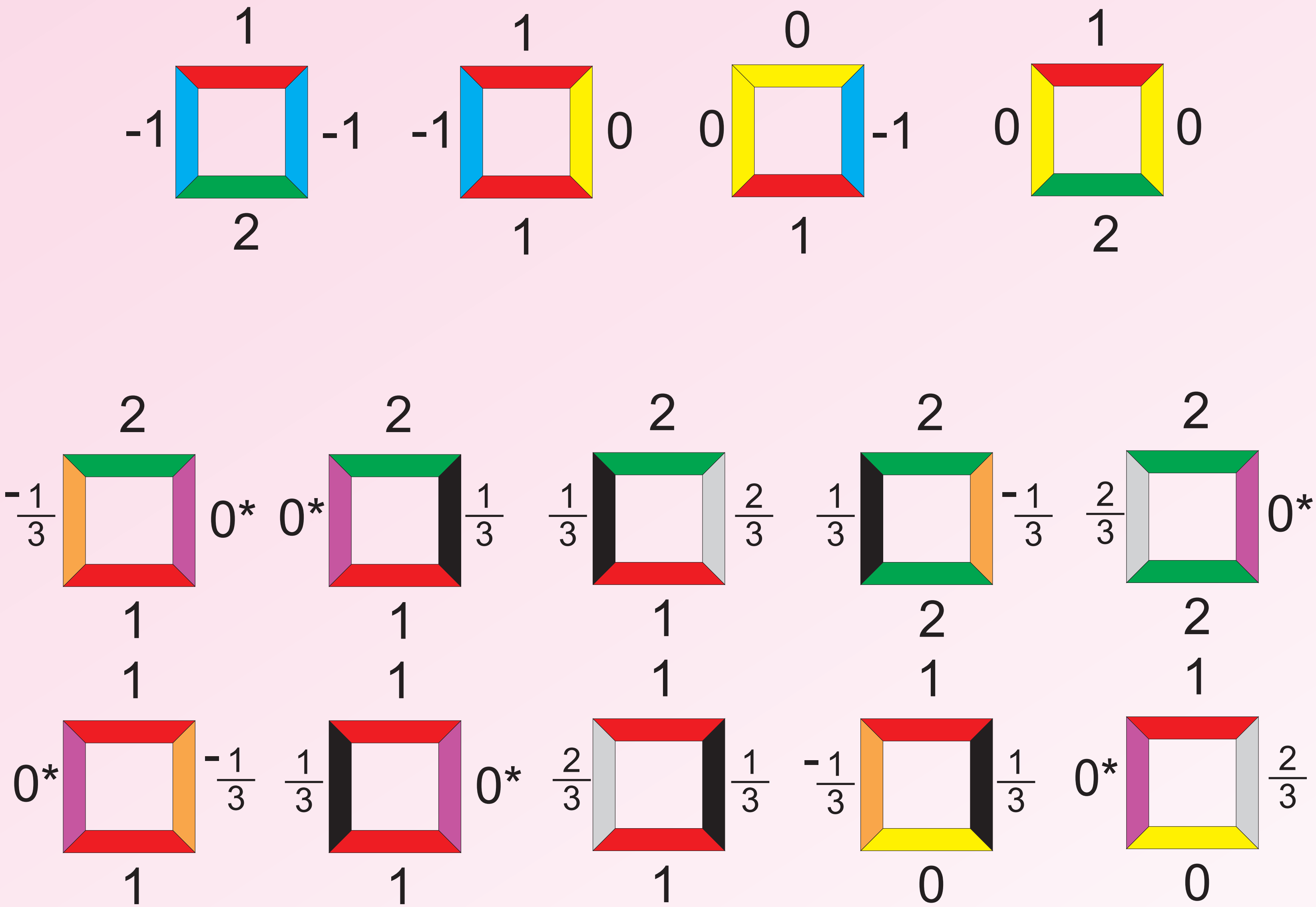
If, moreover, the segment begins and ends in the same color $(w_1 = e_k)$ then

$$\mathbf{q}(\mathbf{n_1} + \mathbf{n_2} + \ldots + \mathbf{n_k}) = (\mathbf{s_1} + \mathbf{s_2} + \ldots + \mathbf{s_k}).$$

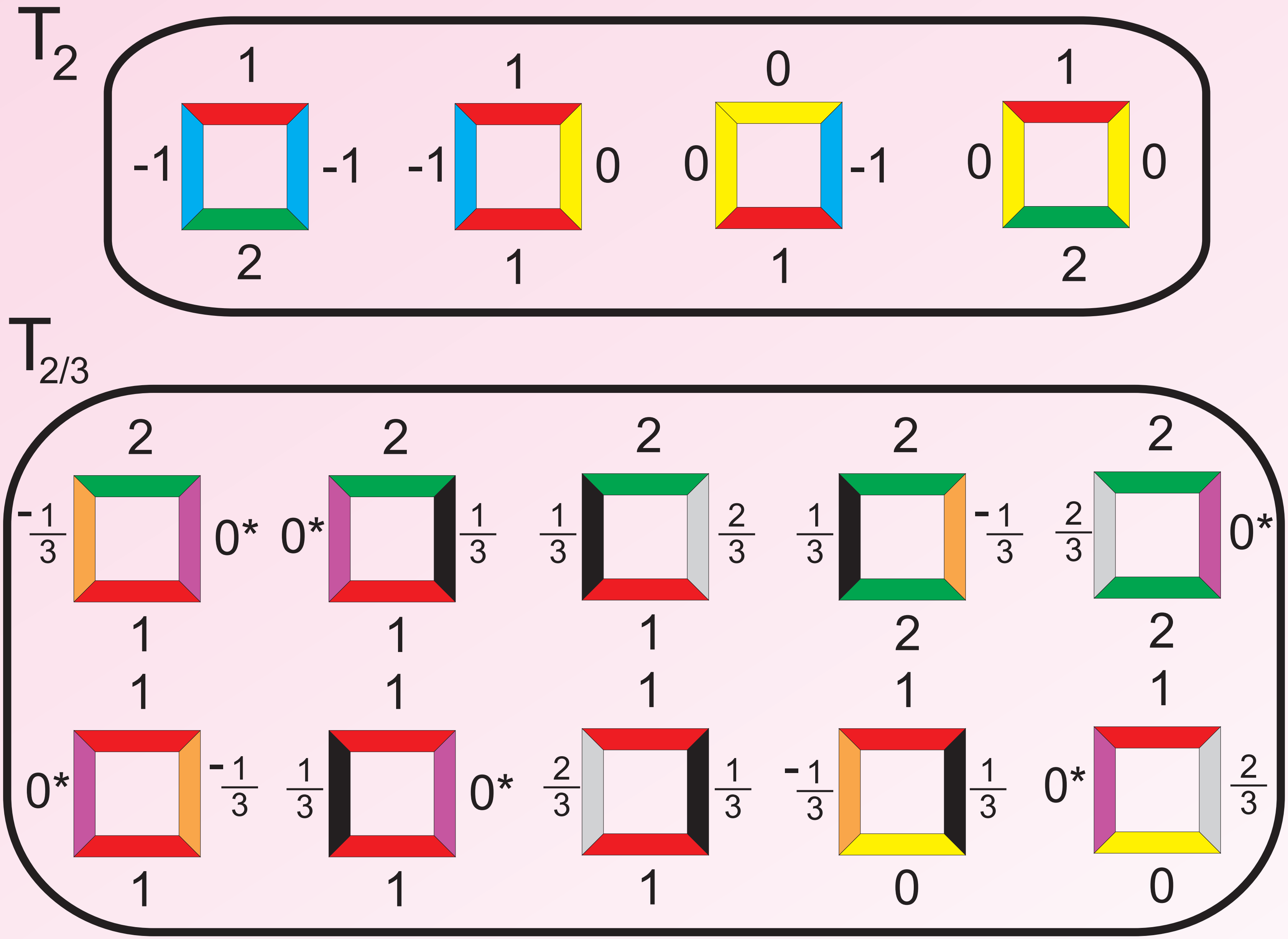
For example, our sample tiles that multiply by $q = 2$ admit the segment



The sum of the bottom labels is twice the sum of the top labels.



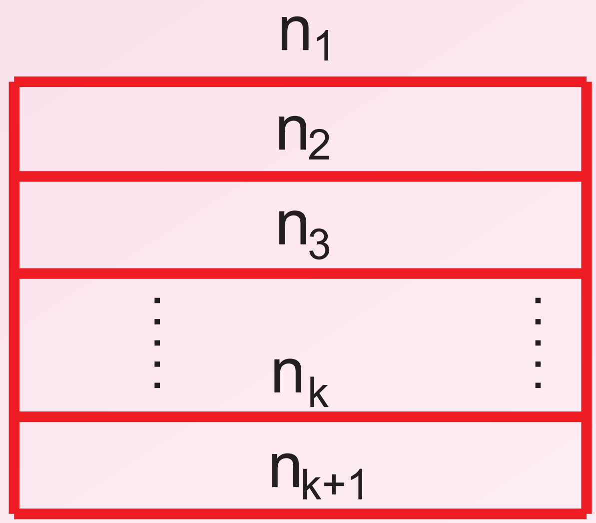
An aperiodic 14 tile set: four tiles that all multiply by 2, and 10 tiles that all multiply by $\frac{2}{3}$.



Let us call these two tile sets T_2 and $T_{2/3}$. Vertical colors are disjoint, so every horizontal row of a tiling comes entirely from one of the two sets.

No periodic tiling exists.

Suppose the contrary: A rectangle can be tiled whose top and bottom rows match and left and right sides match.



Denote by n_i the sum of the numbers on the i'th row. The tiles of the i'th row multiply by $q_i \in \{2, \frac{2}{3}\}$.

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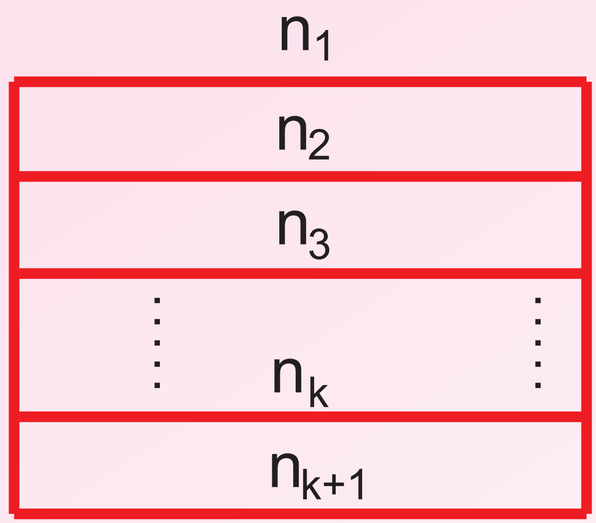

Denote by n_i the sum of the numbers on the i'th row. The tiles of the i'th row multiply by $q_i \in \{2, \frac{2}{3}\}$.

Then $n_{i+1} = q_i n_i$, for all i.

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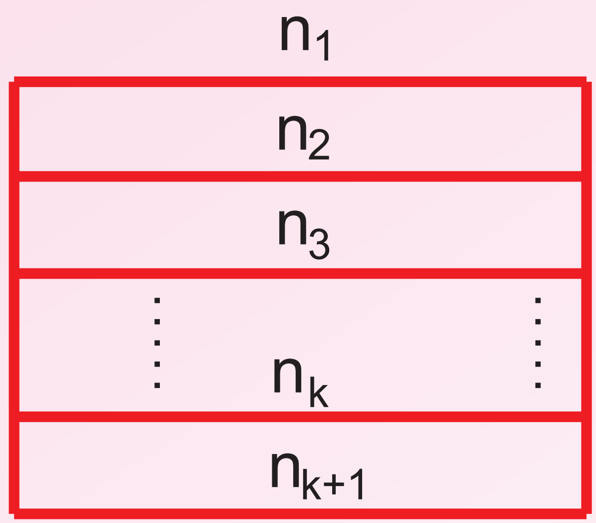

So we have $n_1 q_1 q_2 q_3 \dots q_k = n_{k+1}$

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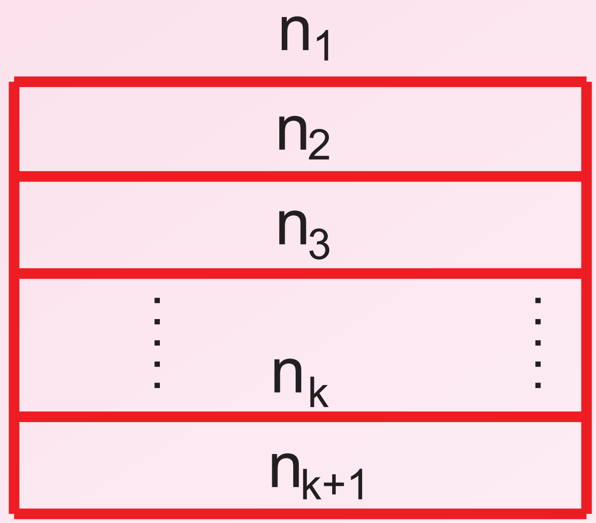

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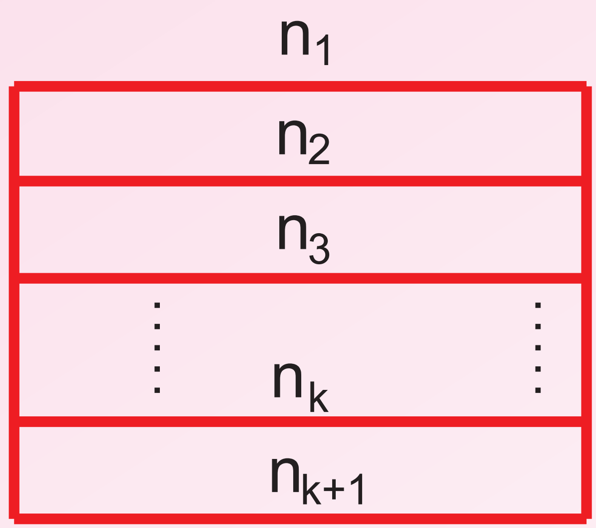

So we have $n_1 q_1 q_2 q_3 \ldots q_k = n_{k+1} = n_1$.

Clearly $n_1 > 0$, so we have $q_1 q_2 q_3 \ldots q_k = 1$.

But this is not possible since 2 and 3 are relatively prime: No product of numbers 2 and $\frac{2}{3}$ can equal 1.

Next step: Proof that a valid tiling of the plane exists.

We use **sturmian** or **balanced** representations of real numbers as bi-infinite sequences of two closest integers.

The representation of any $\alpha \in \mathbb{R}$ is the sequence $B(\alpha)$ whose k'th element is

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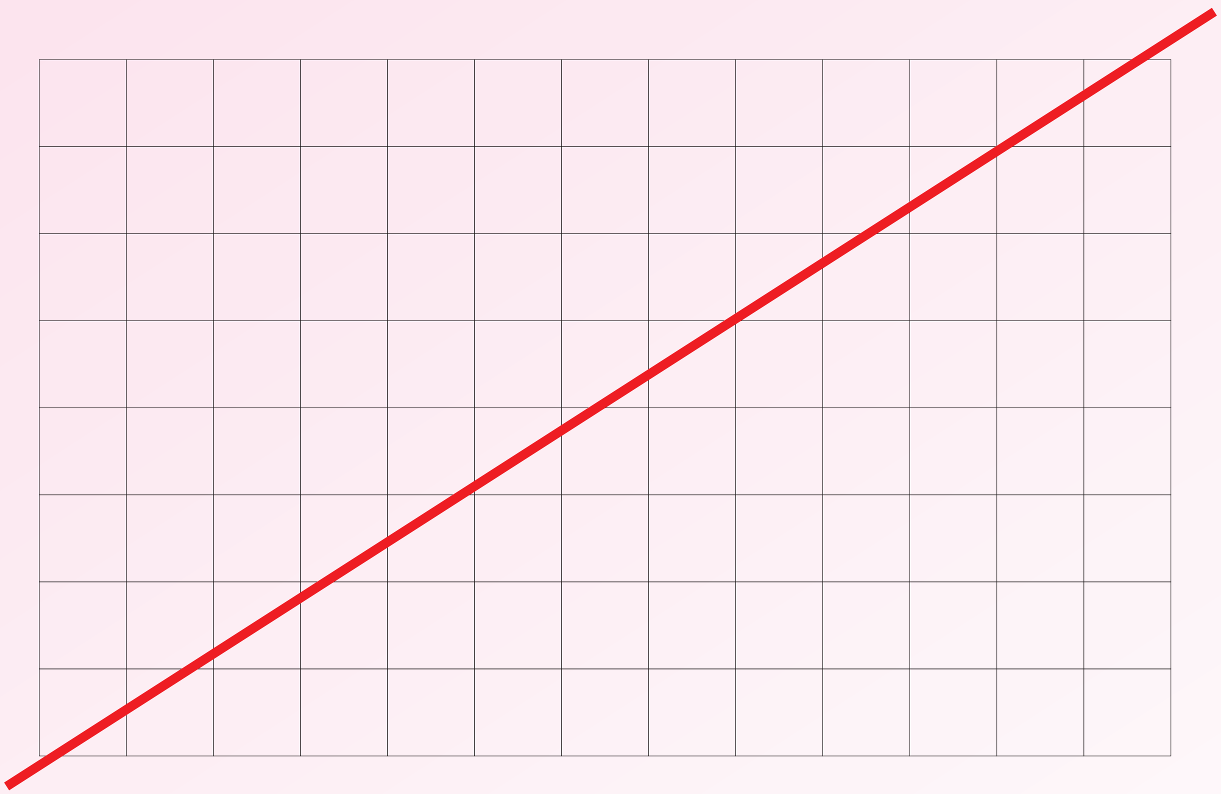

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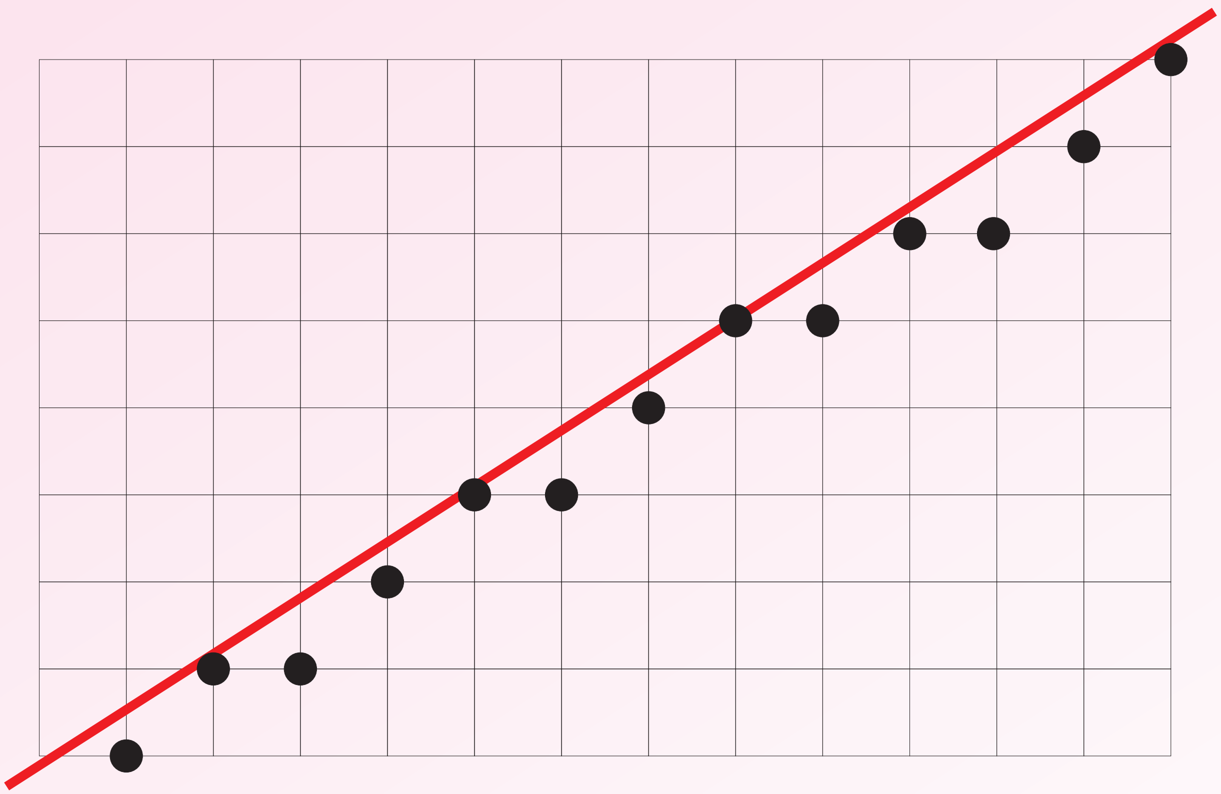

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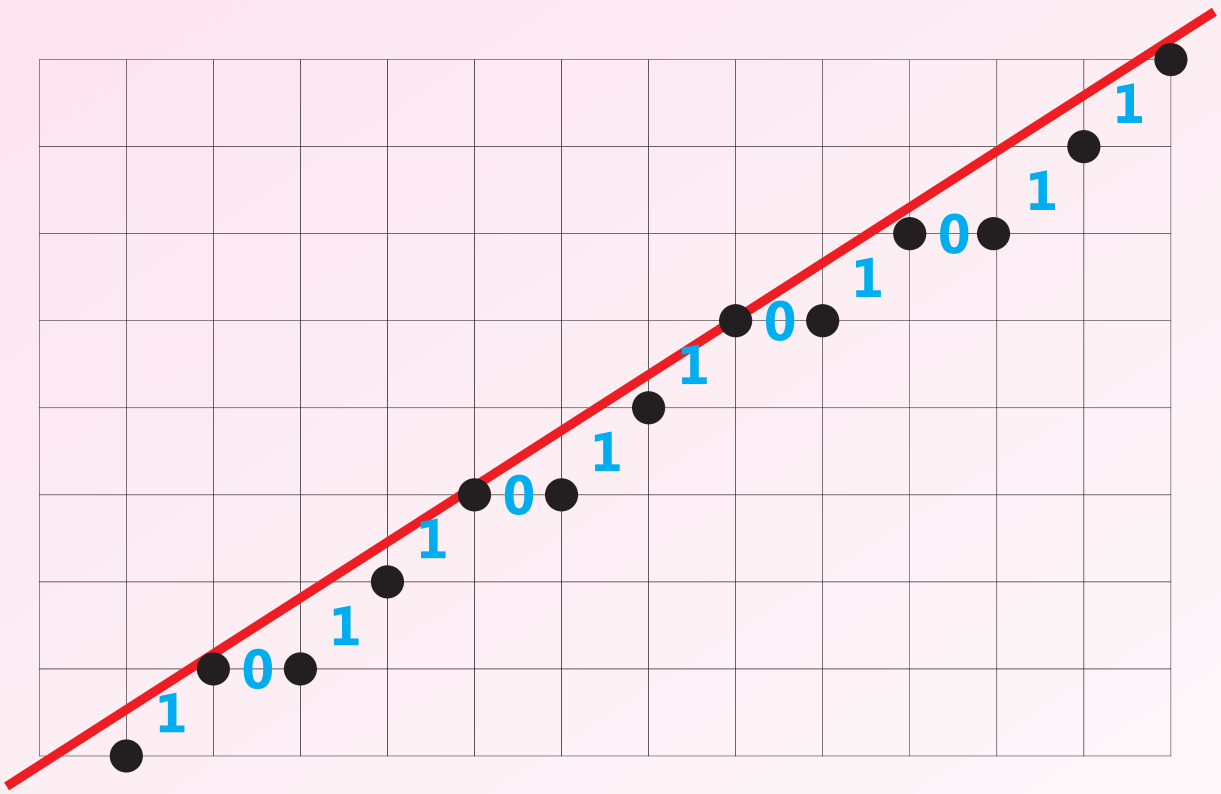

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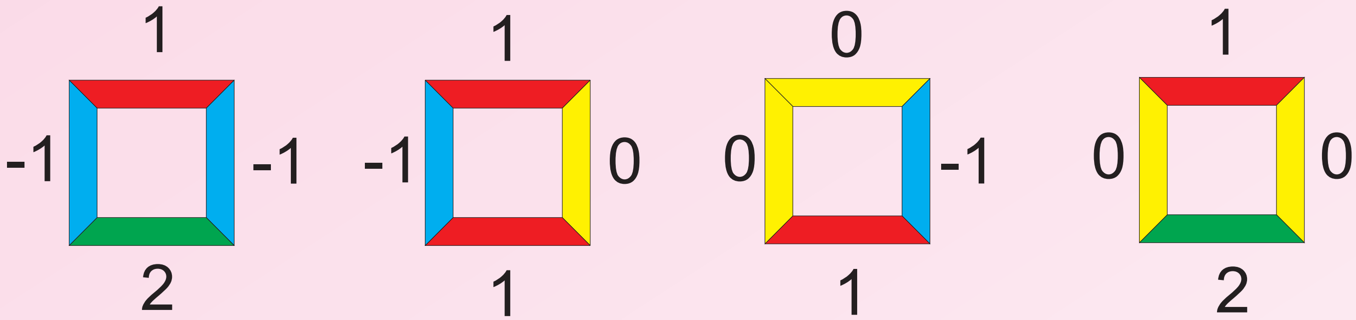
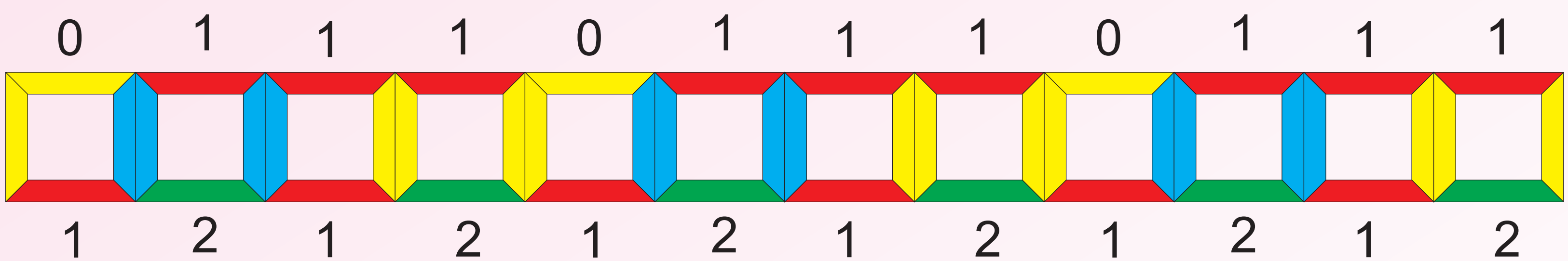
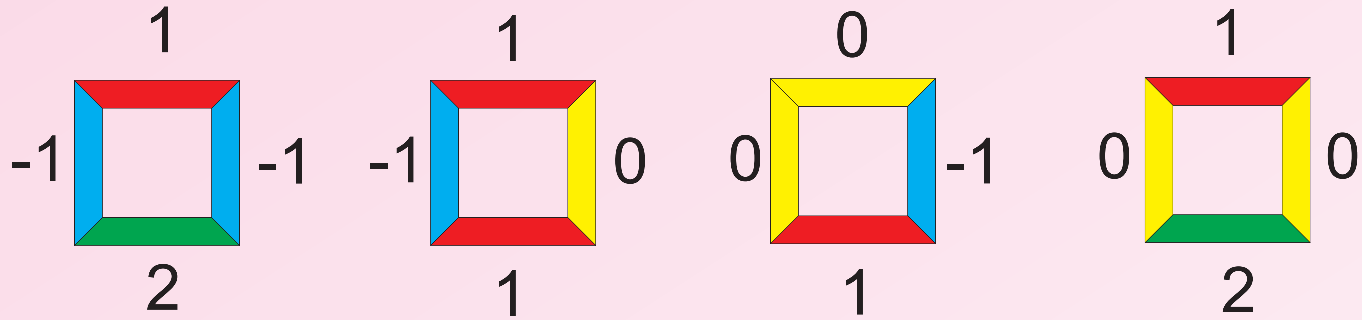


The first tile set T_2 admits a tiling of every infinite horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(2\alpha)$, for all $\alpha \in \mathbb{R}$ satisfying

$$\begin{aligned} 0 &\leq \alpha \leq 1, \text{ and} \\ 1 &\leq 2\alpha \leq 2. \end{aligned}$$

For example, with $\alpha = \frac{3}{4}$:





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$$\left.\begin{array}{ccccl} 0 & \leq & \alpha & \leq & 1, \text{ and} \\ 1 & \leq & 2\alpha & \leq & 2. \end{array}\right\} \iff \frac{1}{2} \leq \alpha \leq 1$$

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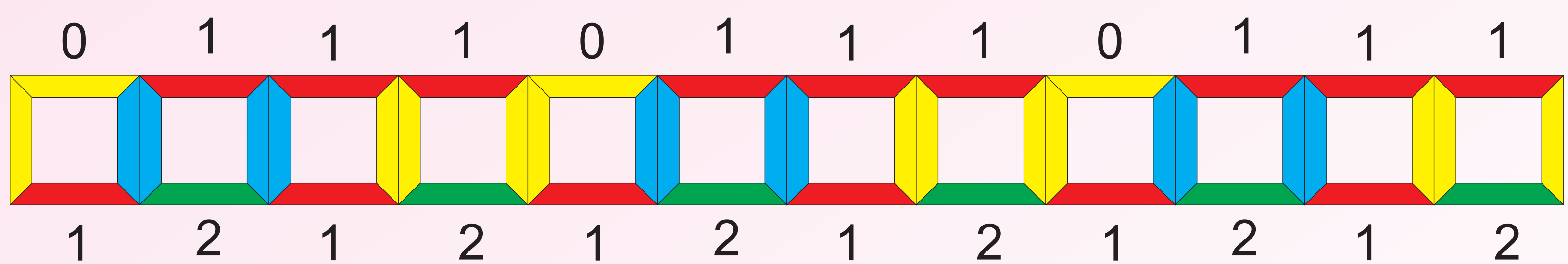
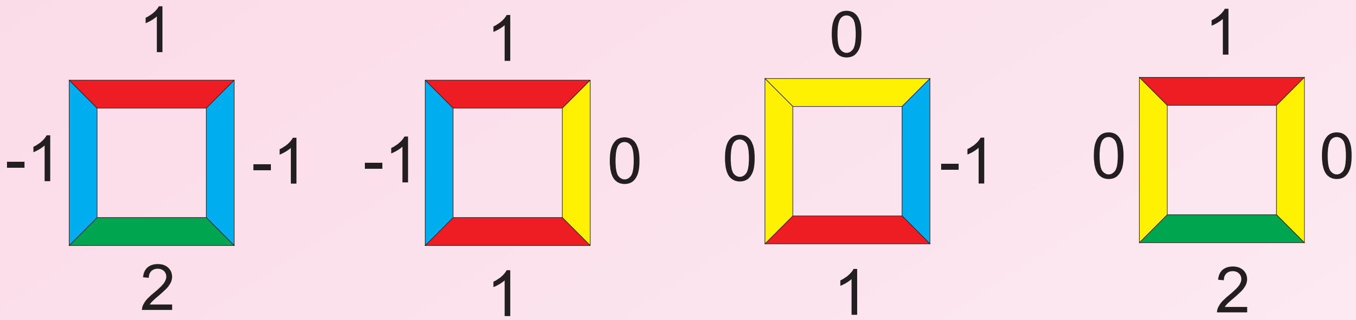
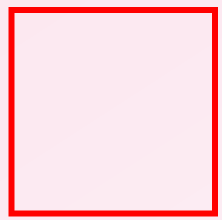




This is guaranteed by including in the tile set for every $\frac{1}{2} \leq \alpha \leq 1$ and every $k \in \mathbb{Z}$ the following tile

$B_k(\alpha)$

$2\lfloor (k-1)\alpha \rfloor - \lfloor 2(k-1)\alpha \rfloor$  $2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$

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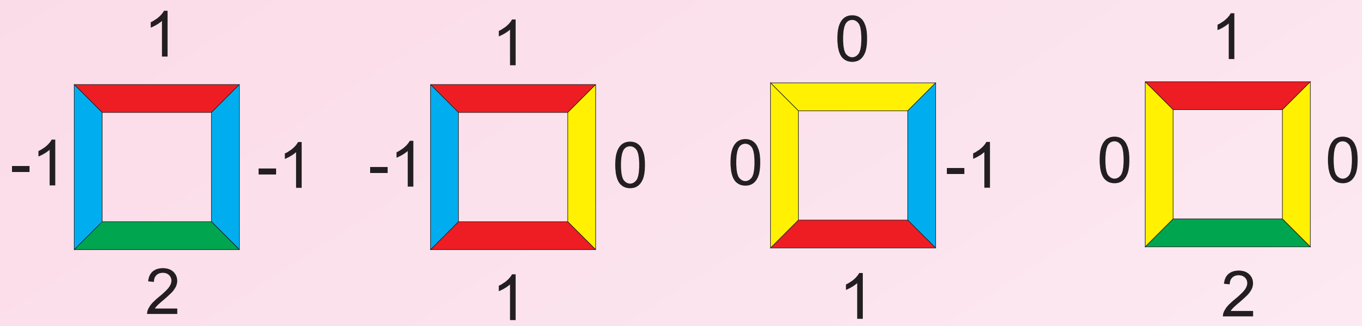

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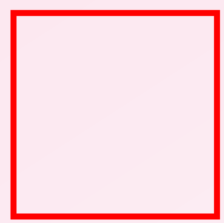
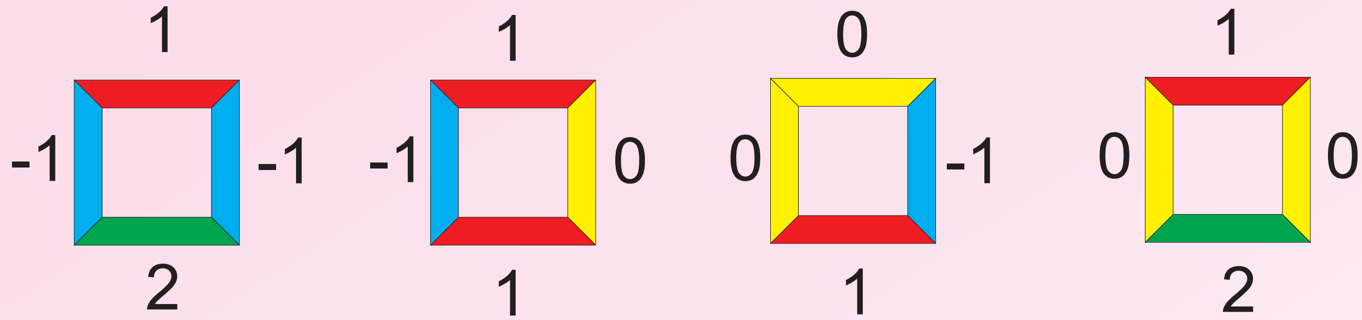


$$2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$

(1) For fixed α the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of α and 2α, respectively.



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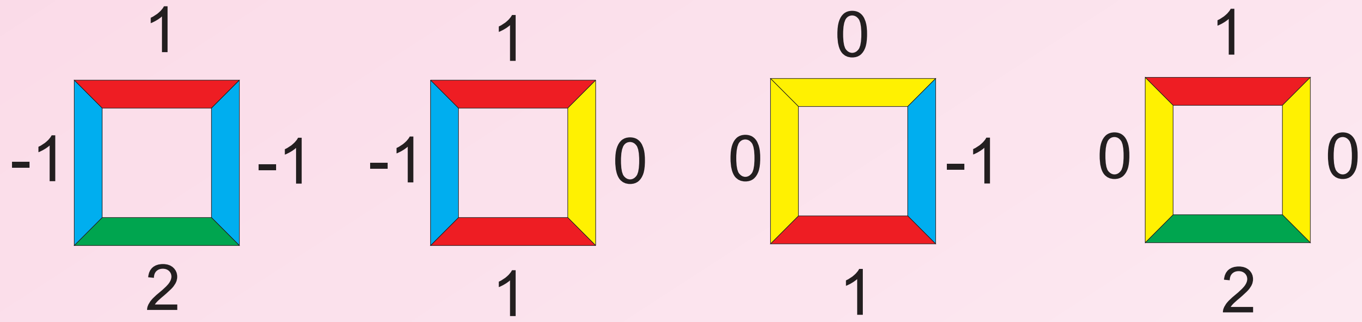
$$B_k(2\alpha)$$

(2) A direct calculation shows that the tile multiplies by 2, that is,

$$2n + w = s + e.$$



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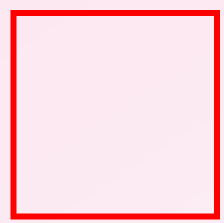

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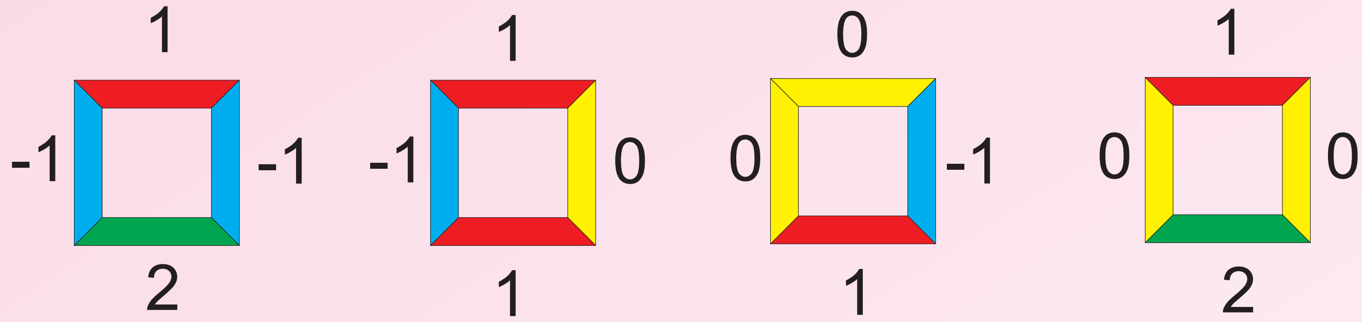
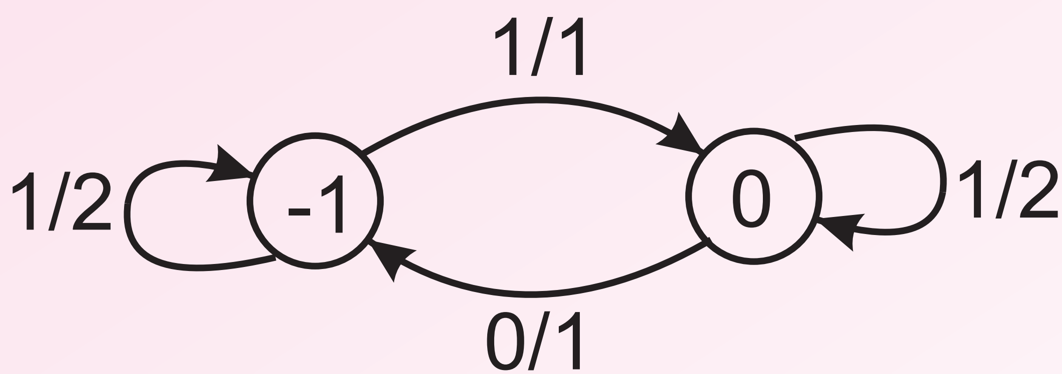
(3) There are only finitely many such tiles, even though there are infinitely many $k \in \mathbb{Z}$ and α. These are the four tiles in T_2.



The four tiles can be also interpreted as edges of a **finite state transducer** whose states are the vertical colors and input/output symbols of transitions are the top and the bottom colors:



A tiling of an infinite horizontal strip is a **bi-infinite path** whose input symbols and output symbols read the top and bottom colors of the strip. We have enough transitions to allow the transducer to convert $B(\alpha)$ into $B(2\alpha)$.

An analogous construction can be done for **any rational multiplier q**. We can construct the following tiles for all $k \in \mathbb{Z}$ and all α in the domain interval:

$$
\begin{array}{ccc}
 & B_k(\alpha) & \\
q\lfloor (k-1)\alpha \rfloor - \lfloor q(k-1)\alpha \rfloor & \boxed{} & q\lfloor k\alpha \rfloor - \lfloor qk\alpha \rfloor \\
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The tiles multiply by q, and they admit a tiling of a horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(q\alpha)$.

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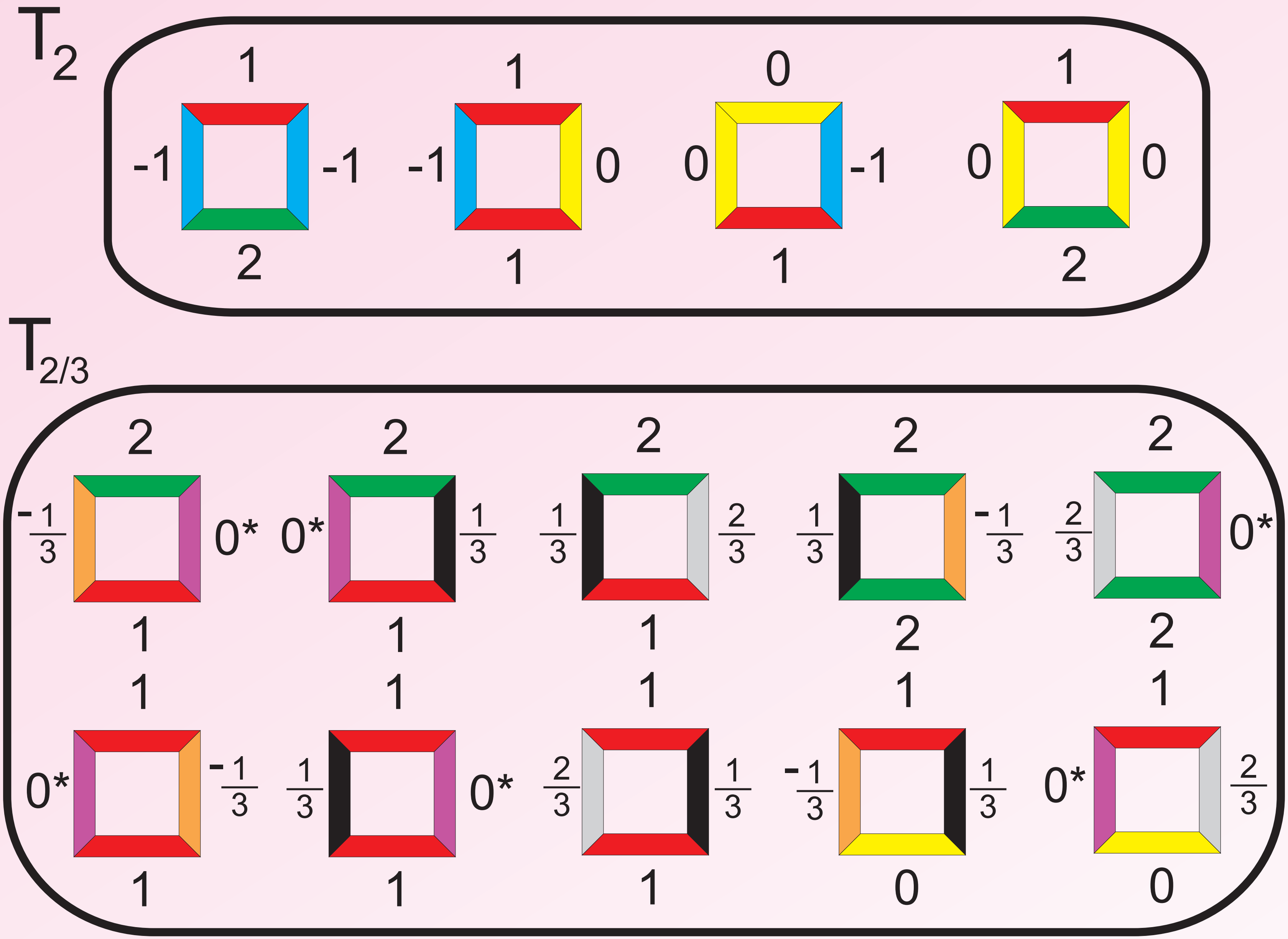
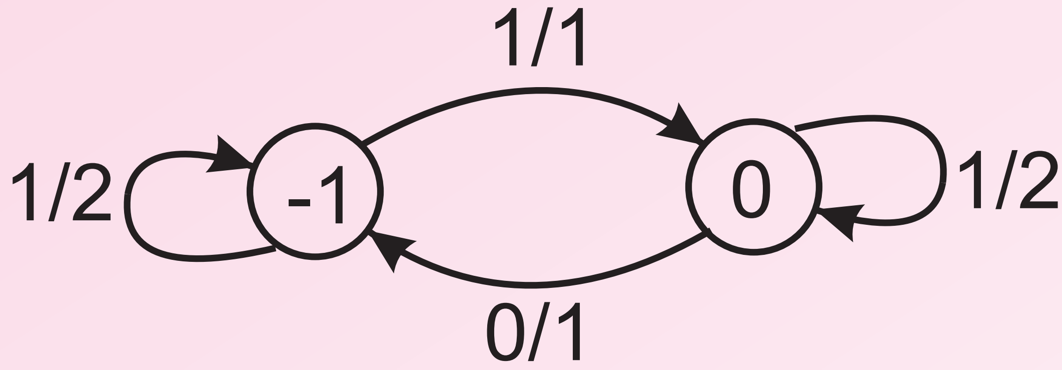
Our second tile set $T_{2/3}$ was constructed in this way for $q = \frac{2}{3}$ and interval $1 \le \alpha \le 2$.



T_2
$T_{2/3}$


1/1
1/2
-1
0
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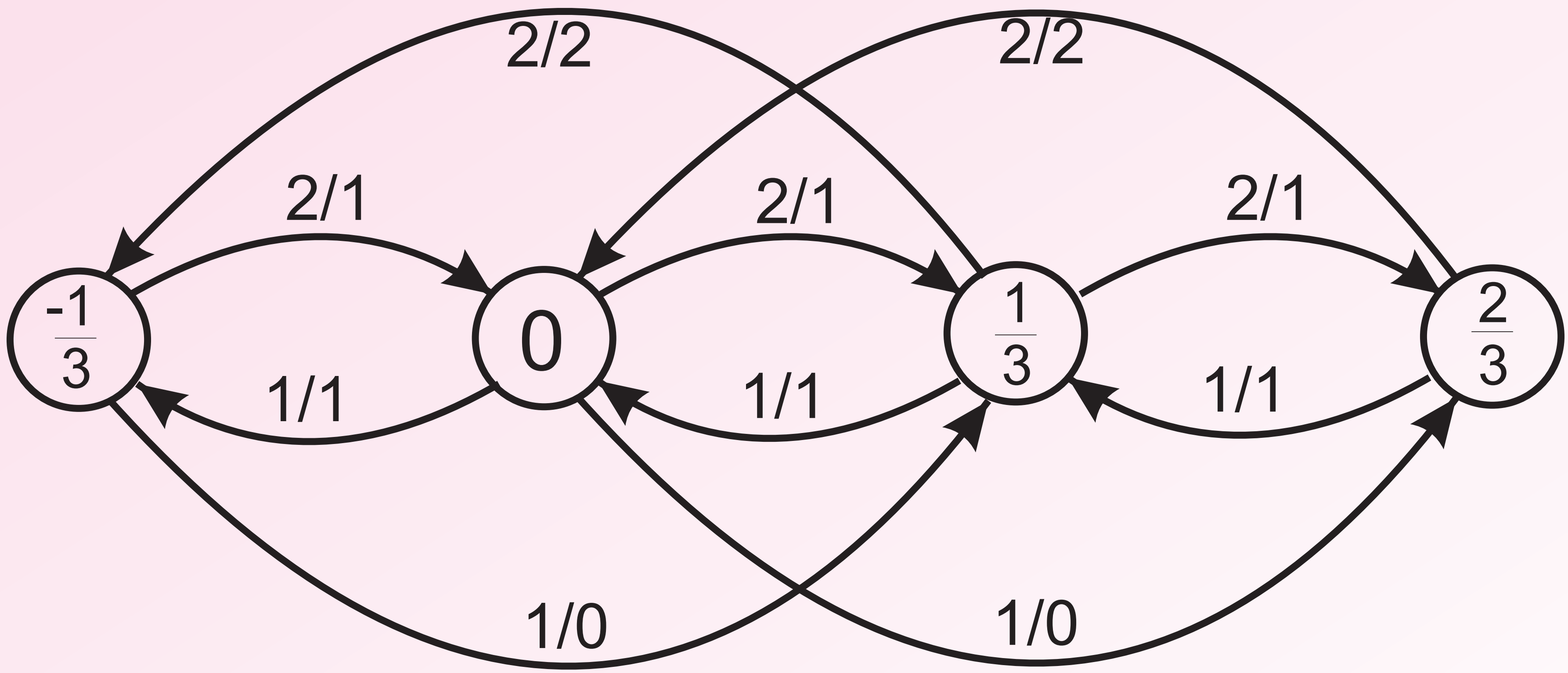
2/2
2/2
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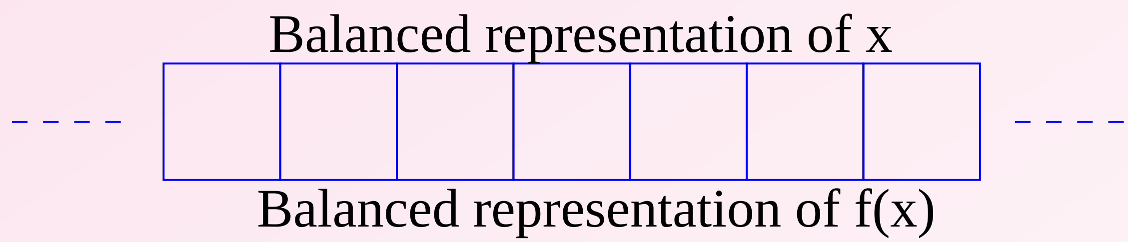
1/0
1/0

The tiles admit valid tilings of the plane that simulate iterations of the piecewise linear dynamical system

$$f : [\frac{1}{2}, 2] \longrightarrow [\frac{1}{2}, 2]$$

where

$$f(x) = \begin{cases} 2x, & \text{if } x \leq 1, \text{ and} \\ \frac{2}{3}x, & \text{if } x > 1. \end{cases}$$



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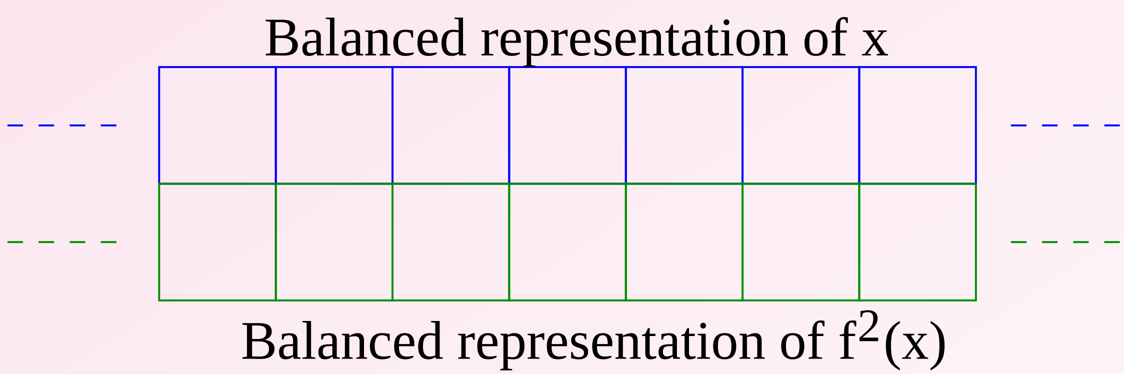

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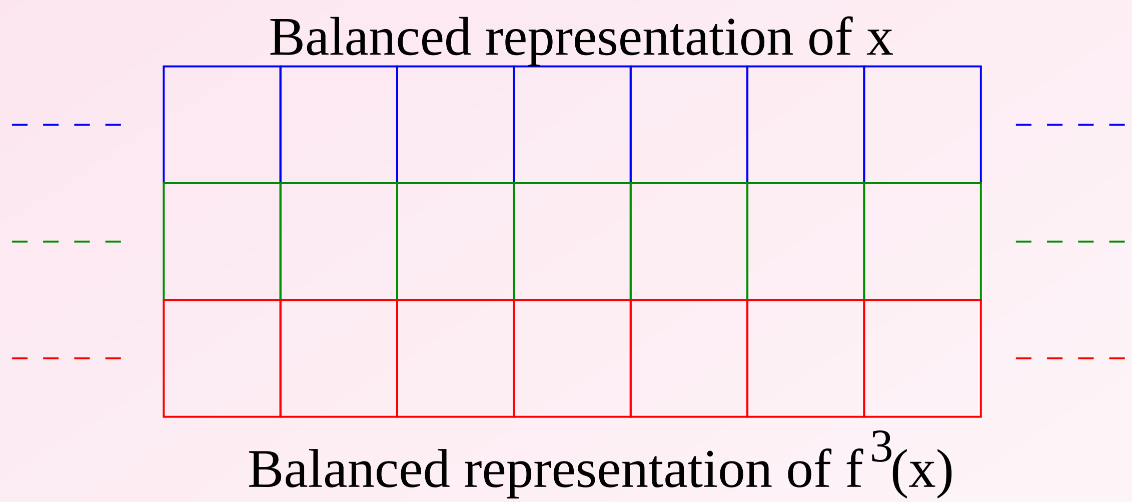

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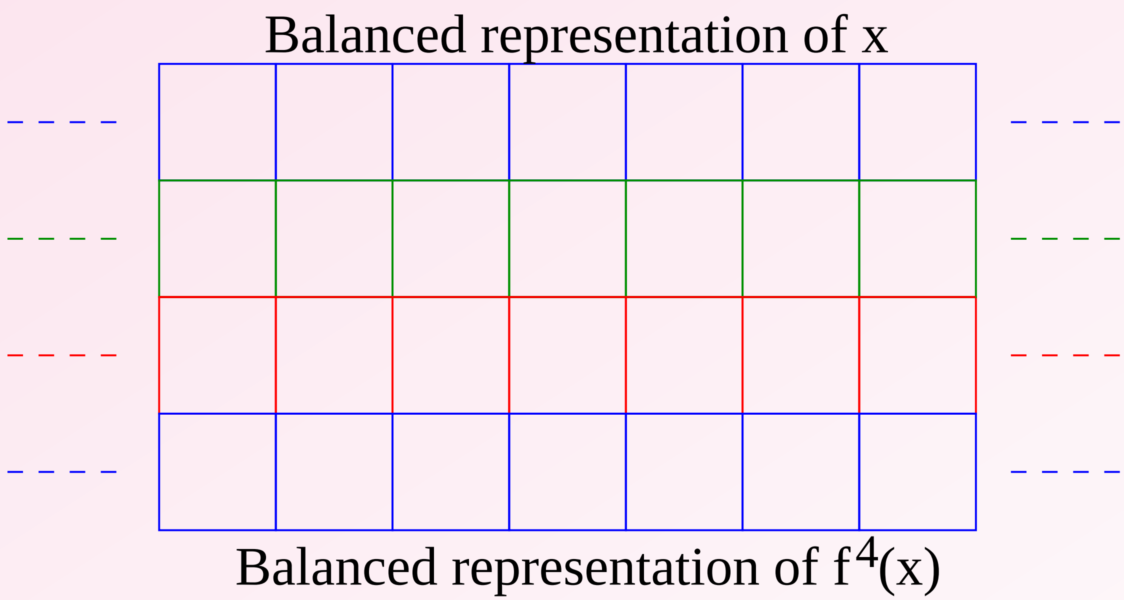

Undecidability of the tiling problem

Similar construction can be effectively carried out for any piecewise linear function on a union of finite intervals of $\mathbb{R}$, as long as the multiplications are with rational numbers q.

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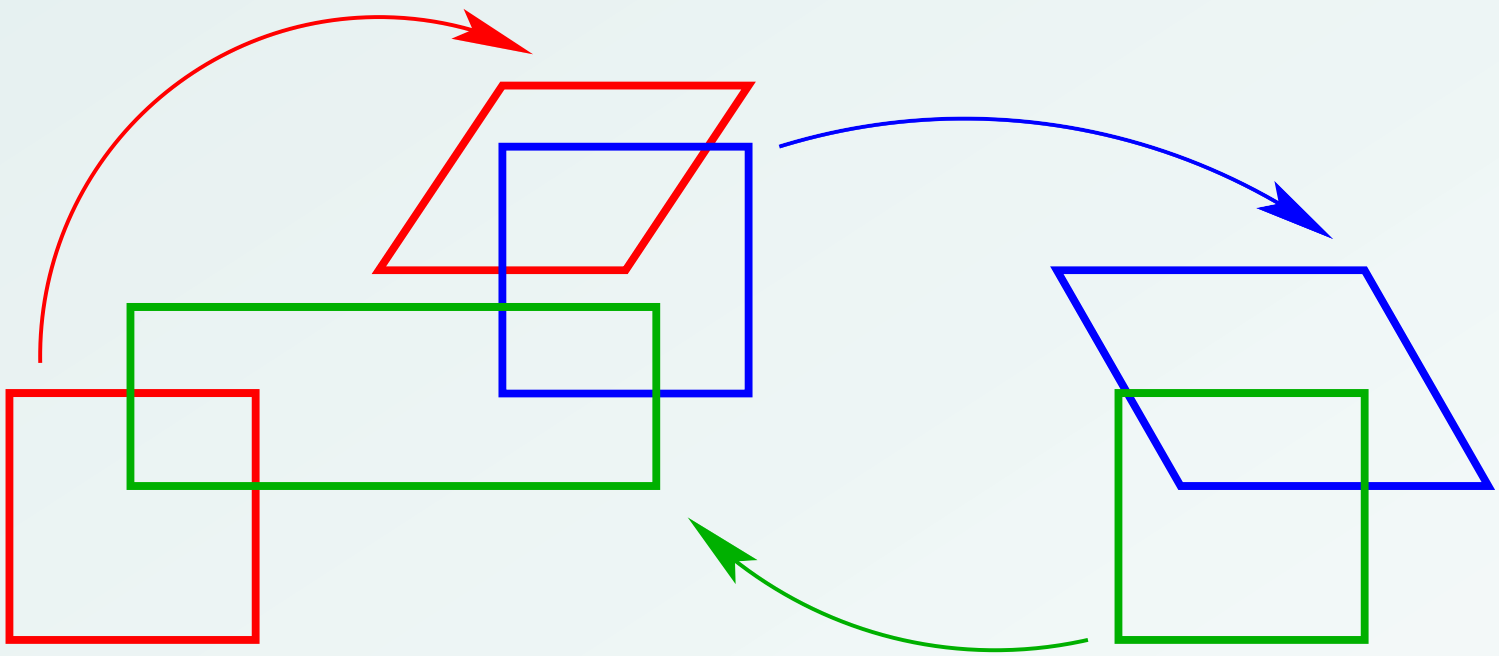
In order to prove **undecidability results** concerning tilings we want to simulate more complex dynamical systems that can carry out Turing computations.

We generalize the construction in two ways:

- from linear maps to affine maps, and
- from $\mathbb{R}$ to $\mathbb{R}^2$, (or $\mathbb{R}^d$ for any d).

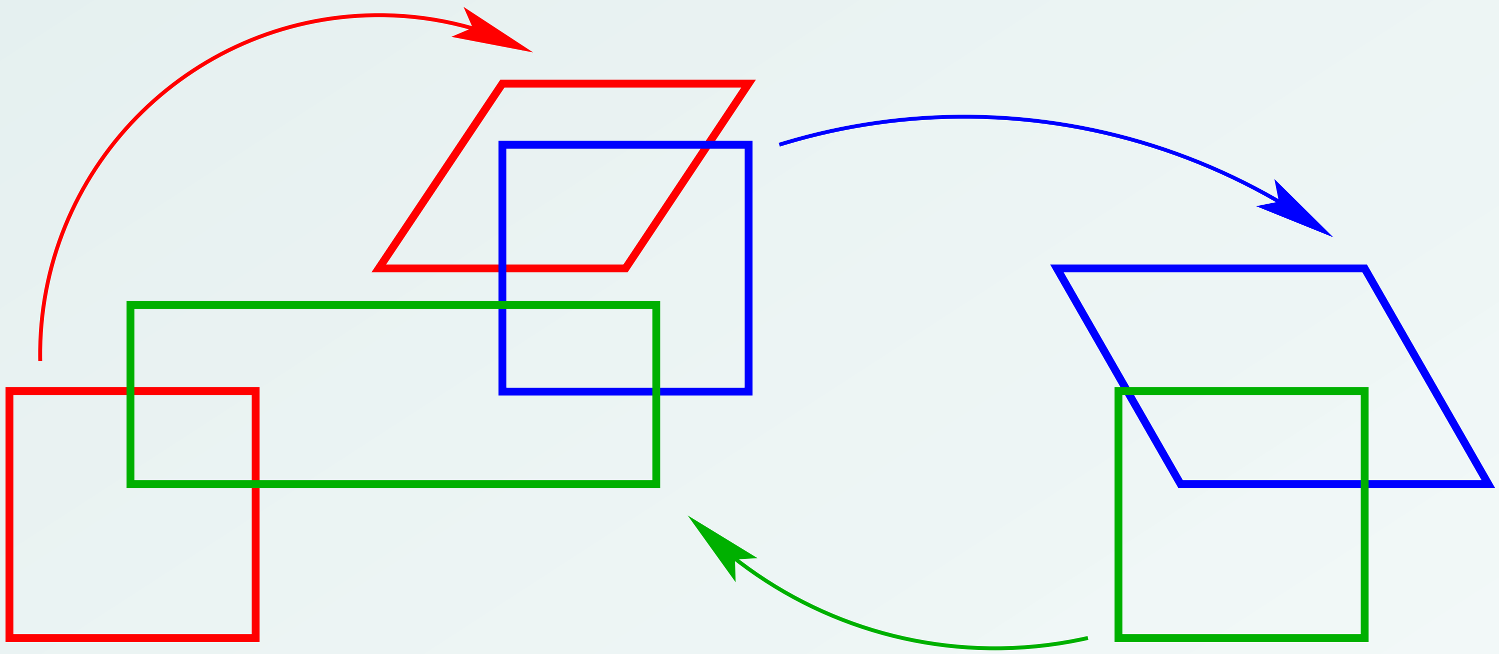
Immortality of piecewise affine maps



Consider a system of finitely many pairs (U_i, f_i) where

- U_i are disjoint unit squares of the plane with integer corners,
- f_i are affine transformations with rational coefficients.

Square U_i serves as the domain where f_i may be applied.



The system determines a function

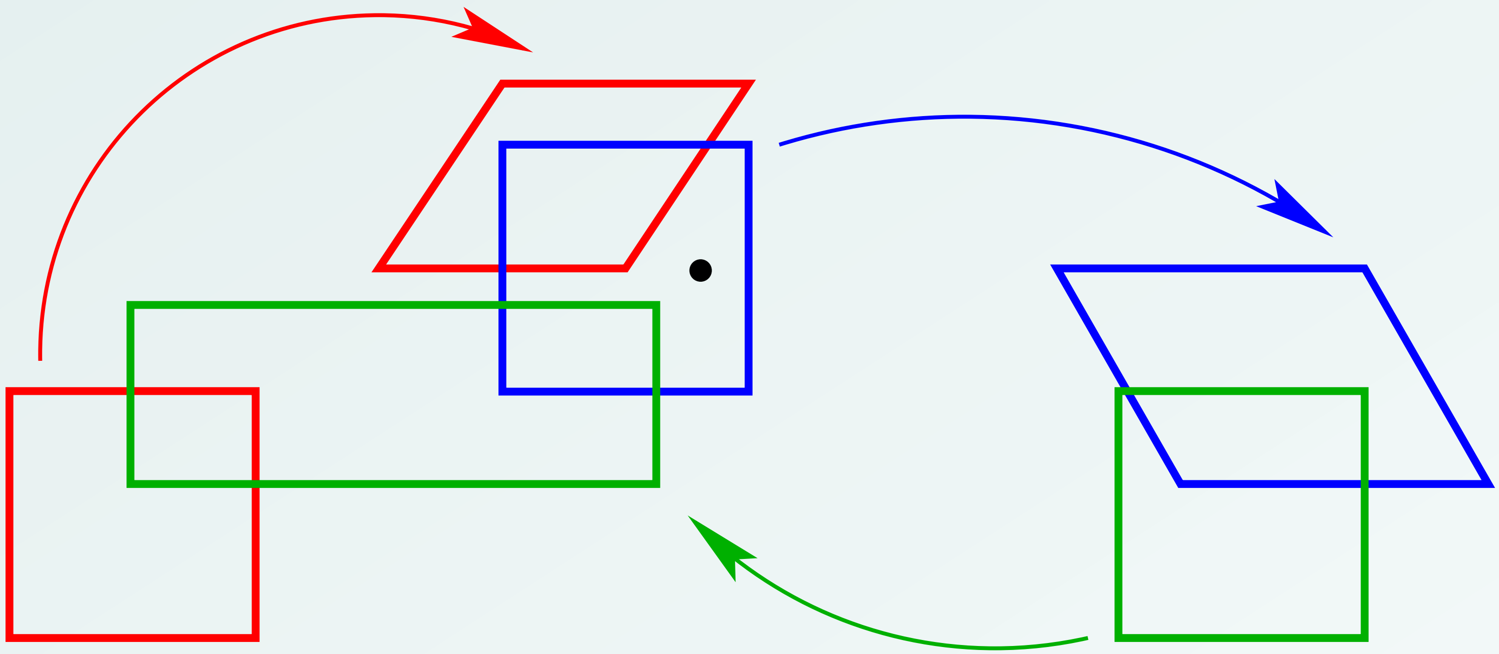
$$f : D \longrightarrow \mathbb{R}^2$$

whose domain is

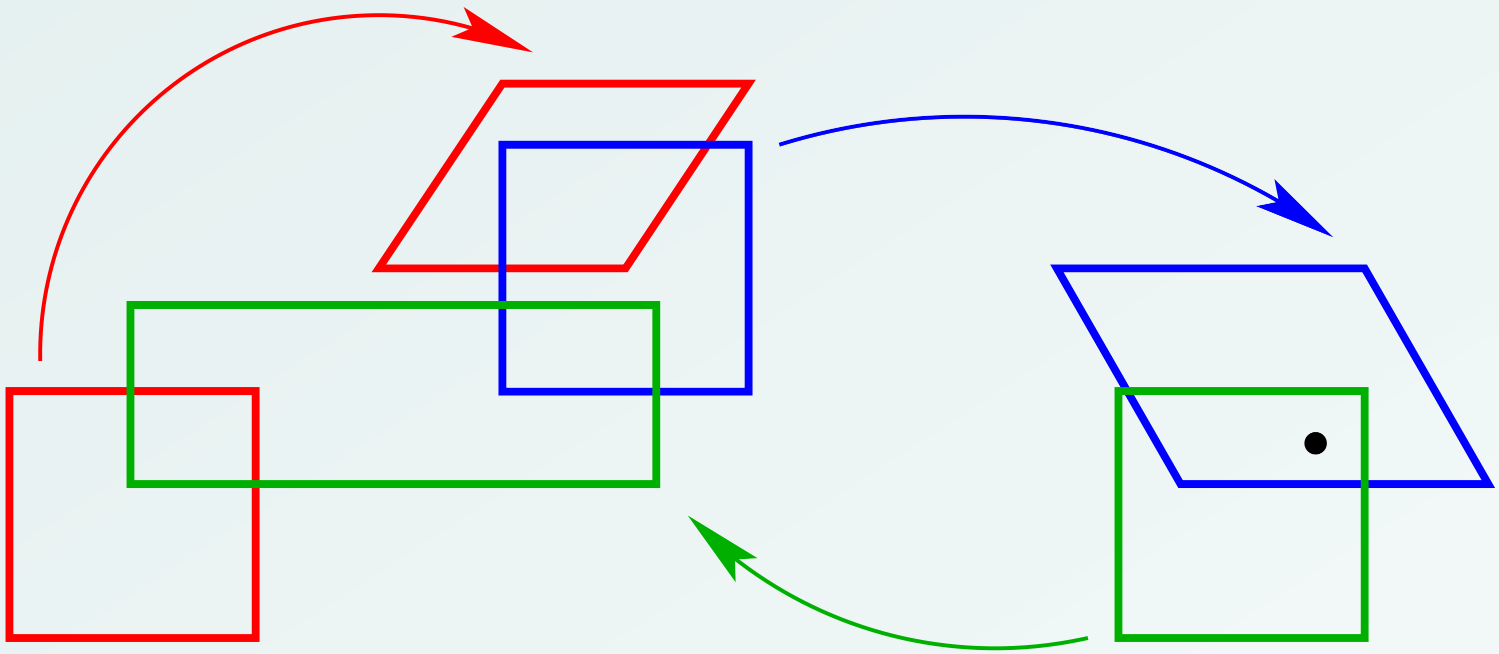
$$D = \bigcup_i U_i$$

and

$$f(\vec{x}) = f_i(\vec{x}) \text{ for all } \vec{x} \in U_i.$$



The orbit of $\vec{x} \in D$ is the iteration of f starting at point $\vec{x}$. The iteration can be continued as long as the point remains in the domain D.



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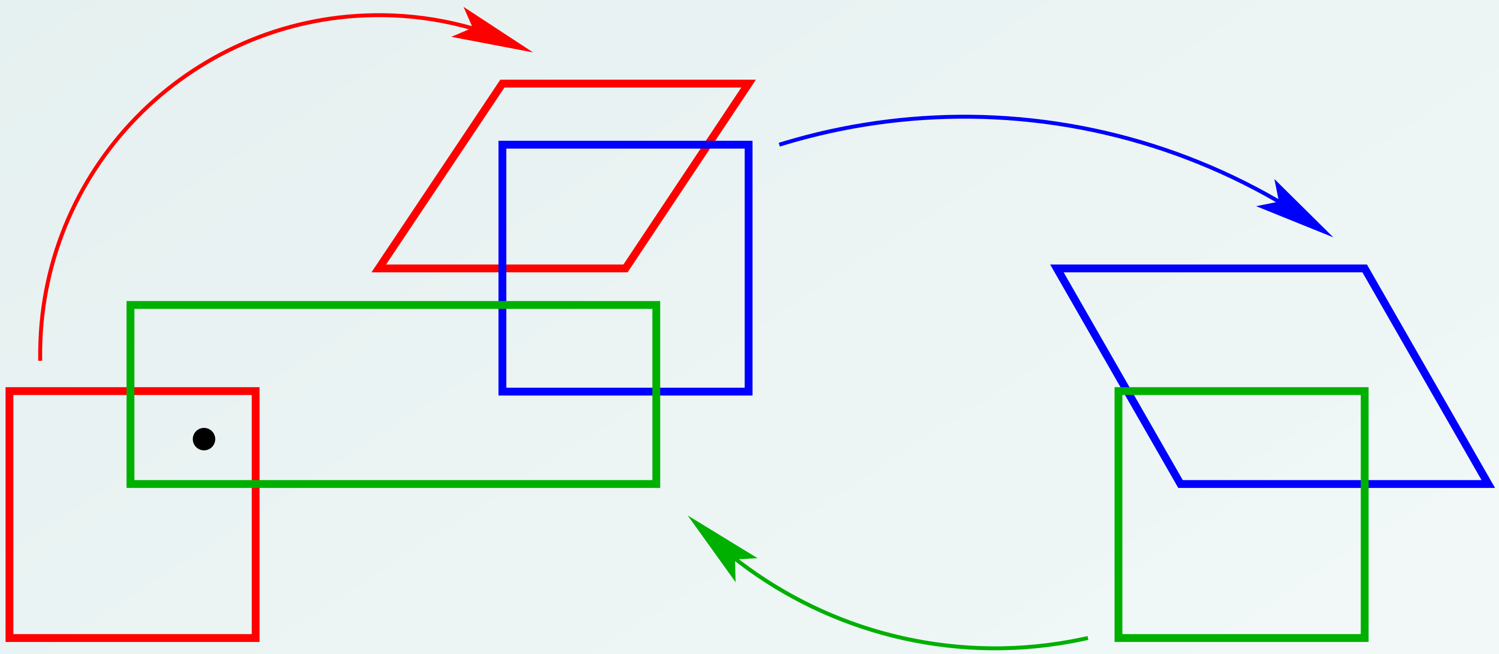

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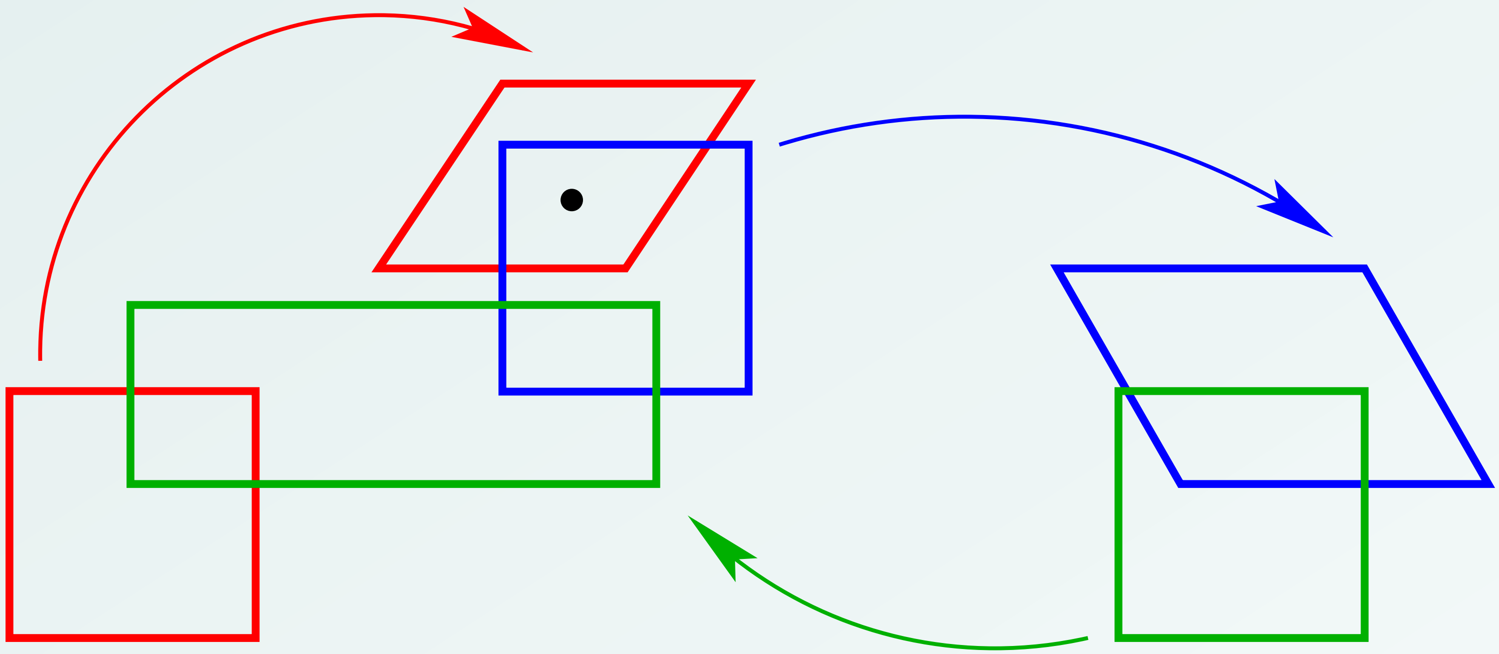

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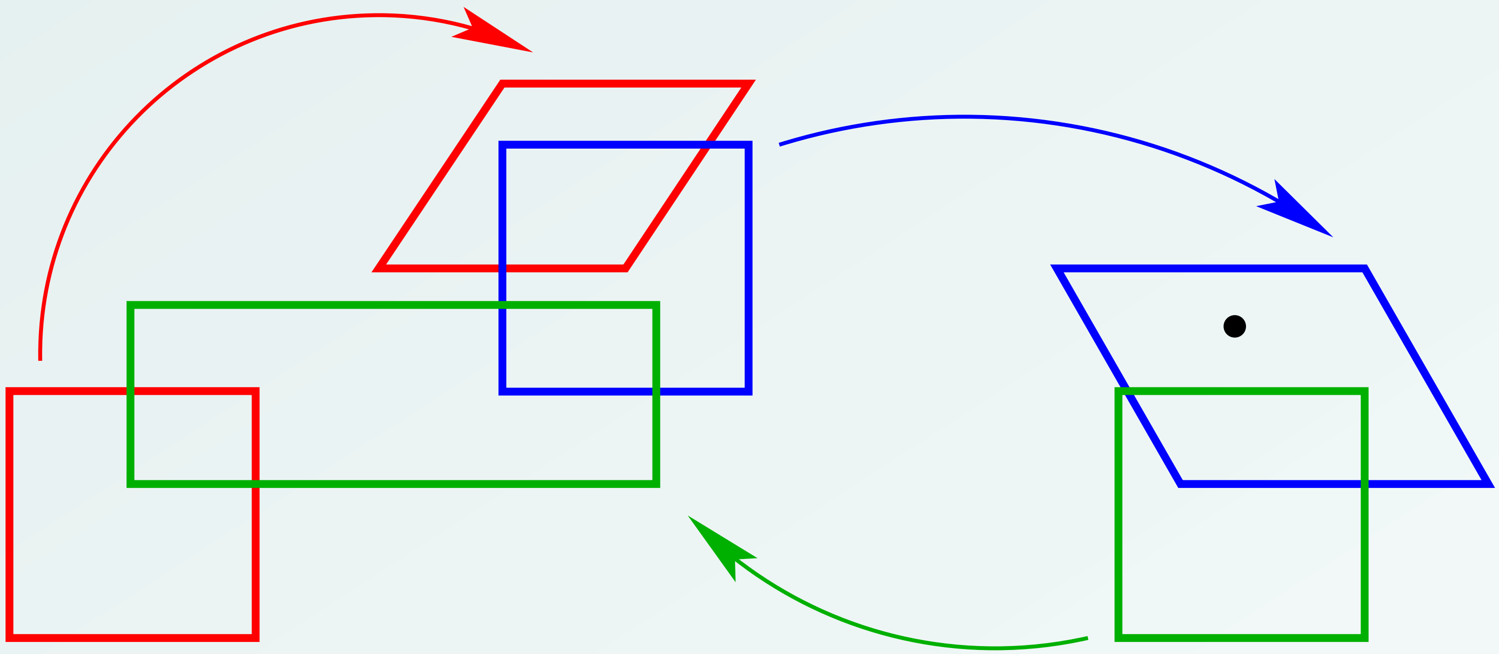
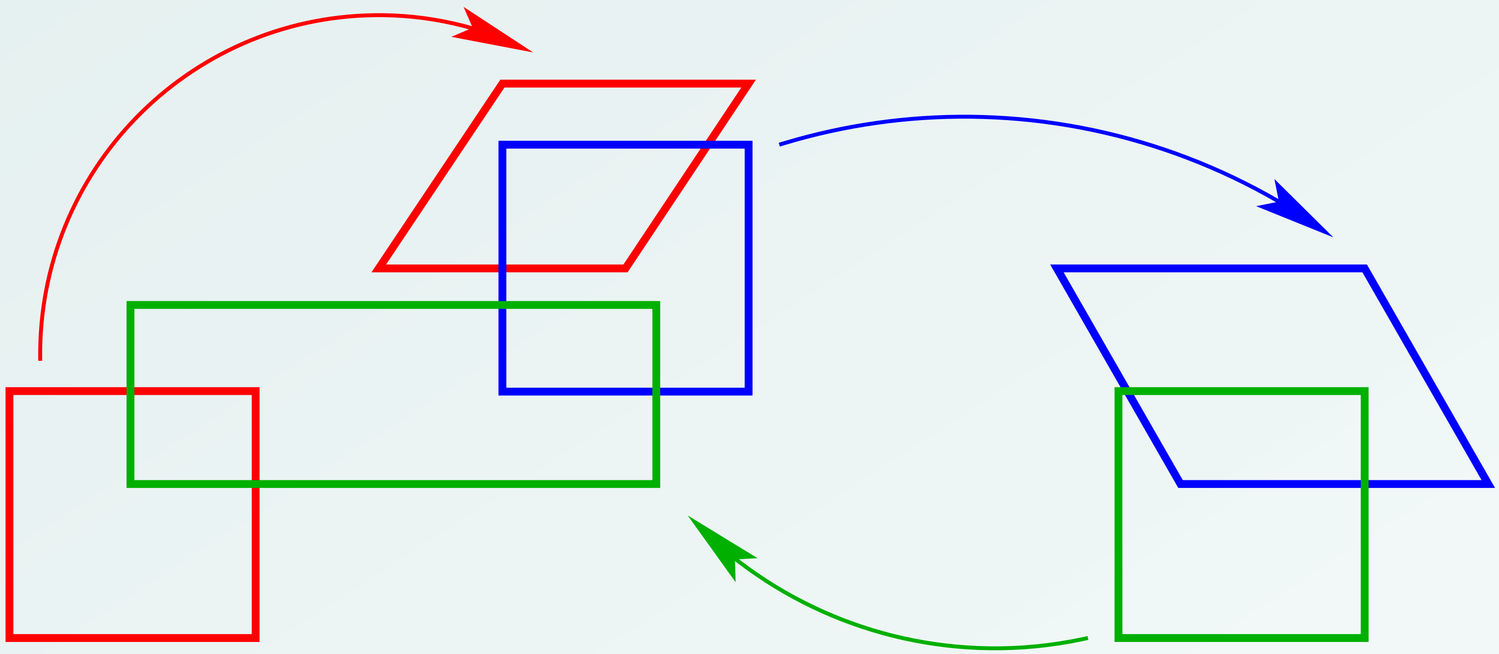


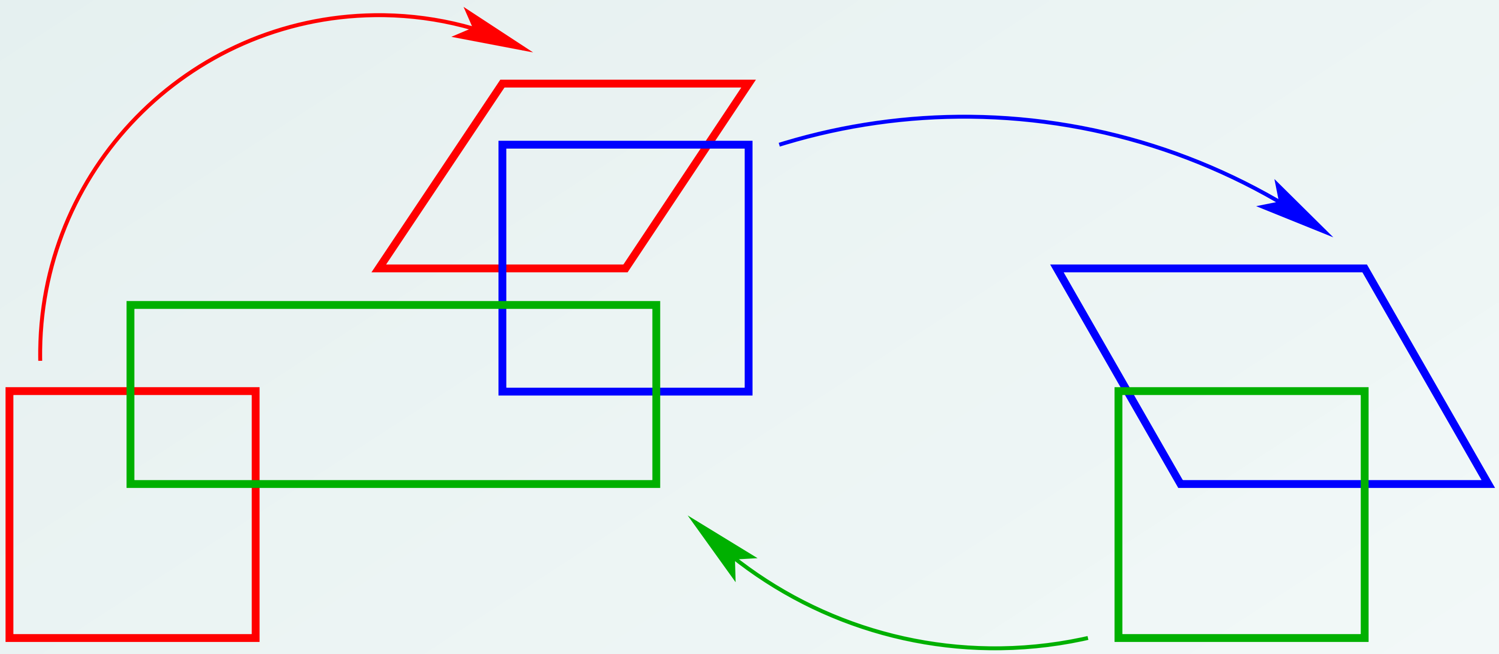
But if the point goes outside of the domain, the system **halts**.

If the iteration always halts, regardless of the starting point $\vec{x}$, the system is **mortal**. Otherwise it is **immortal**: there is an immortal point $\vec{x} \in D$ from which a non-halting orbit begins.



Immortality problem: Is a given system of affine maps immortal?

Proposition: The immortality problem is undecidable.



Immortality problem: Is a given system of affine maps immortal?

Proposition: The immortality problem is undecidable.

Follows from a standard simulation of **Turing machines** by two-dimensional piecewise affine transformations, and from:

Theorem (Hooper 1966): It is undecidable if a given Turing machine has any immortal configurations.

Next: We effectively construct Wang tiles that are forced to simulate iterations of given piecewise affine maps.

Then the undecidability of the tiling problem follows: a valid tiling exists if and only if the dynamical system has an infinite orbit (which is undecidable).

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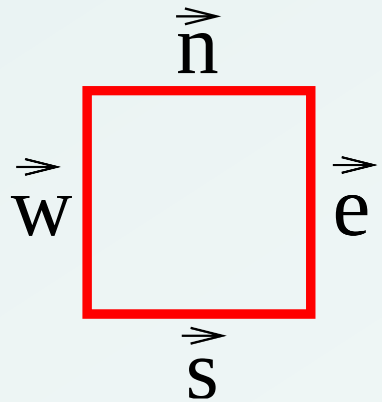
The construction is very similar to the earlier construction of 14 aperiodic tiles.

The colors in our Wang tiles are elements of $\mathbb{R}^2$.

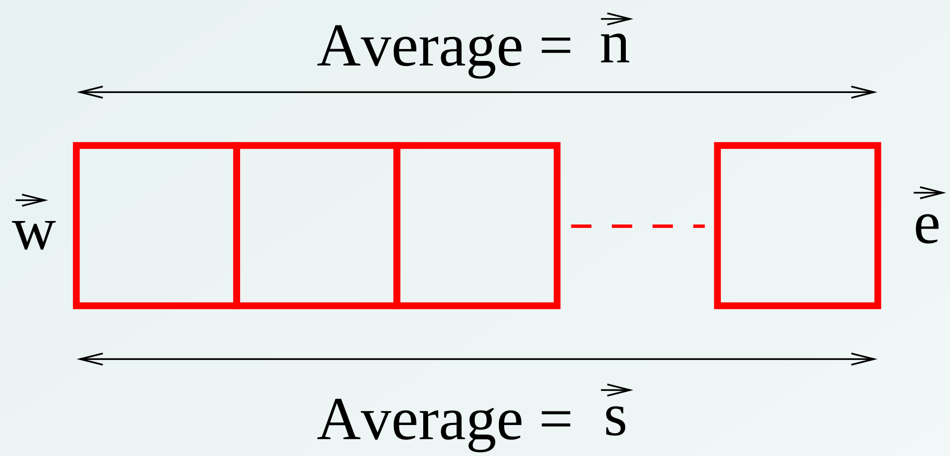
Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an affine function. We say that tile



computes function f if

$$f(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

Suppose we have a correctly tiled horizontal segment of length n where all tiles compute the same f.



It easily follows that

$$f(\vec{n}) + \frac{1}{n}\vec{w} = \vec{s} + \frac{1}{n}\vec{e},$$

where $\vec{n}$ and $\vec{s}$ are the averages of the top and the bottom labels.

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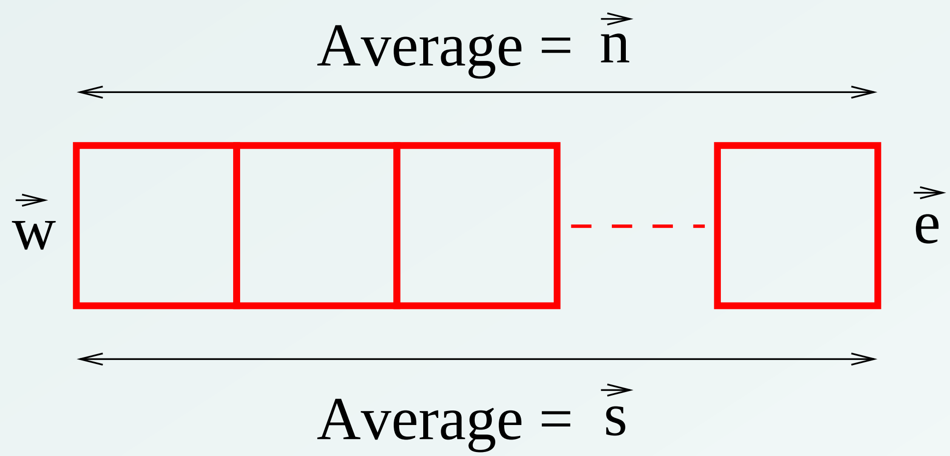

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$$f(\vec{n}) + \frac{1}{n}\vec{w} = \vec{s} + \frac{1}{n}\vec{e},$$

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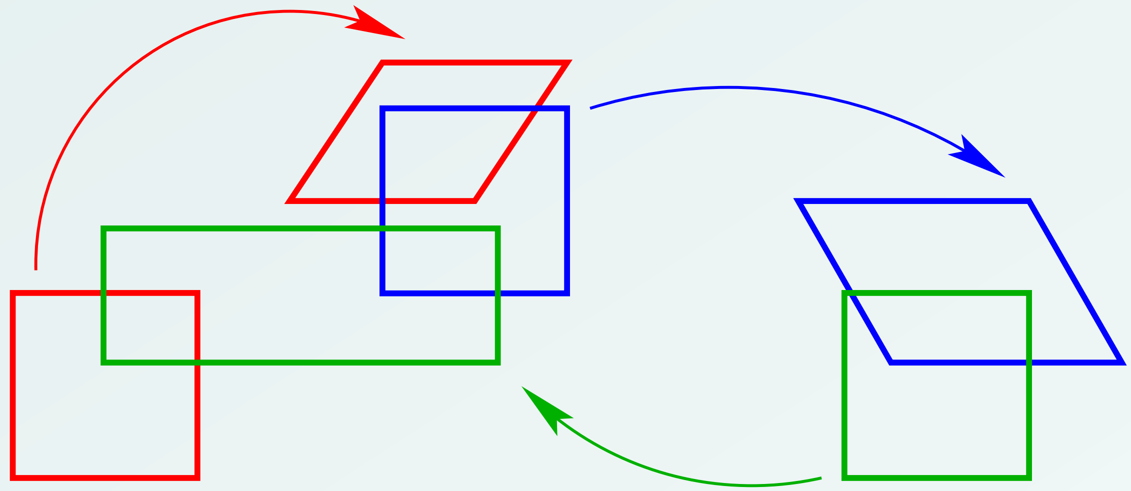
As the segment is made longer, the effect of the carry-in and carry-out labels $\vec{w}$ and $\vec{e}$ vanish.



Consider a system of affine maps f_i and unit squares U_i.

For each i we construct a set T_i of Wang tiles

- that compute function f_i, and
- whose top edge labels $\vec{n}$ are in U_i.

We also make sure that tiles of different sets T_i and T_j cannot be mixed on any horizontal row of tiles. Let

$$T = \bigcup_i T_i.$$

Claim: If such T admits a valid tiling then the system of affine maps has an immortal point.

Indeed: An immortal point is obtained as the average of the top labels on a horizontal row of the tiling. The averages on subsequent horizontal rows below are the iterates of that point under the dynamical system.

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Indeed: An immortal point is obtained as the average of the top labels on a horizontal row of the tiling. The averages on subsequent horizontal rows below are the iterates of that point under the dynamical system.

Small technicality: If the average over an infinite horizontal row does not exist then we take an accumulation point of averages of finite segments instead. . . this always exists.

We still have to detail how to choose the tiles so that also the **converse** is true: any immortal orbit of the affine maps gives a valid tiling.

The tile set corresponding to a rational affine map

$$f_i(\vec{x}) = M\vec{x} + \vec{b}$$

and its domain square U_i consists of all tiles

$$\begin{array}{ccc}
 & B_k(\vec{x}) & \\
\begin{array}{r} f_i(\lfloor (k-1)\vec{x} \rfloor) \\ -\lfloor (k-1) f_i(\vec{x}) \rfloor \\ +(k-1)\vec{b} \end{array} & & \begin{array}{r} f_i(\lfloor k\vec{x} \rfloor) \\ -\lfloor k f_i(\vec{x}) \rfloor \\ +k\vec{b} \end{array} \\
 & B_k(f_i(\vec{x})) &
\end{array}$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

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\begin{array}{ccc}
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\end{array}
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where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(1) For fixed $\vec{x} \in U_i$ the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of $\vec{x}$ and $f_i(\vec{x})$, respectively.

$$
\begin{array}{ccc}
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 & B_k(f_i(\vec{x})) &
\end{array}
$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(2) A direct calculation shows that the tile computes function f_i, that is,

$$f_i(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

$$B_k(\vec{x})$$

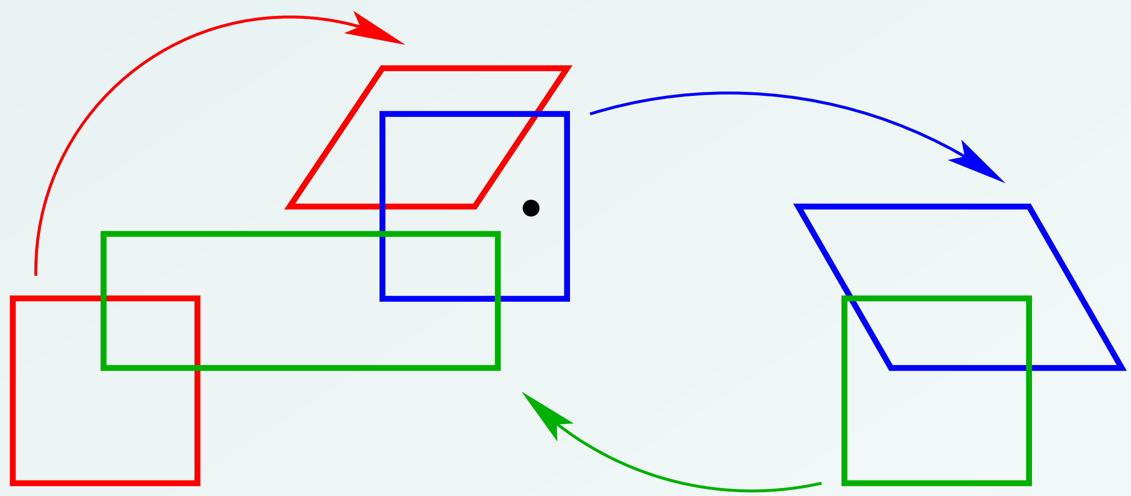
$$\begin{array}{ccc} f_i(\lfloor (k-1)\vec{x} \rfloor) & & f_i(\lfloor k\vec{x} \rfloor) \\ -\lfloor (k-1) f_i(\vec{x}) \rfloor & \boxed{} & -\lfloor k f_i(\vec{x}) \rfloor \\ +(k-1)\vec{b} & & +k\vec{b} \end{array}$$

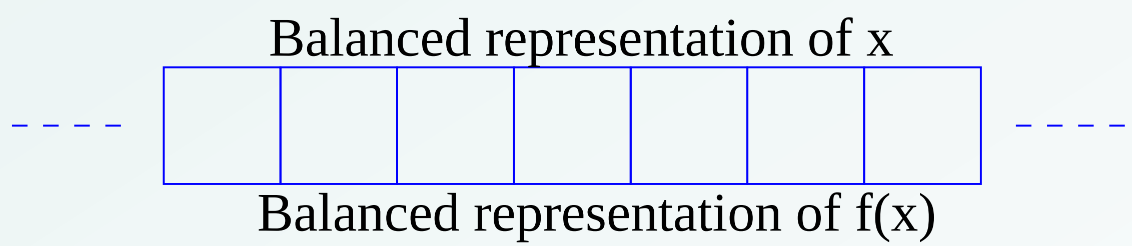
$$B_k(f_i(\vec{x}))$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(3) Because f_i is rational, there are only finitely many such tiles (even though there are infinitely many $k \in \mathbb{Z}$ and $\vec{x} \in U_i$). The tiles can be effectively constructed.

If there is an infinite orbit then a tiling exists where the labels of the horizontal rows read the balanced representations of the points of the orbit:





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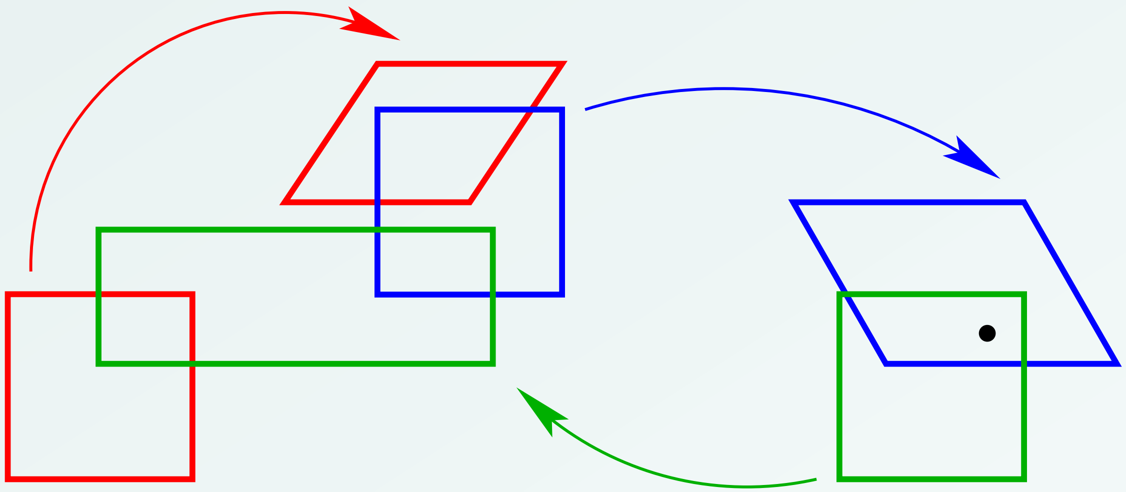

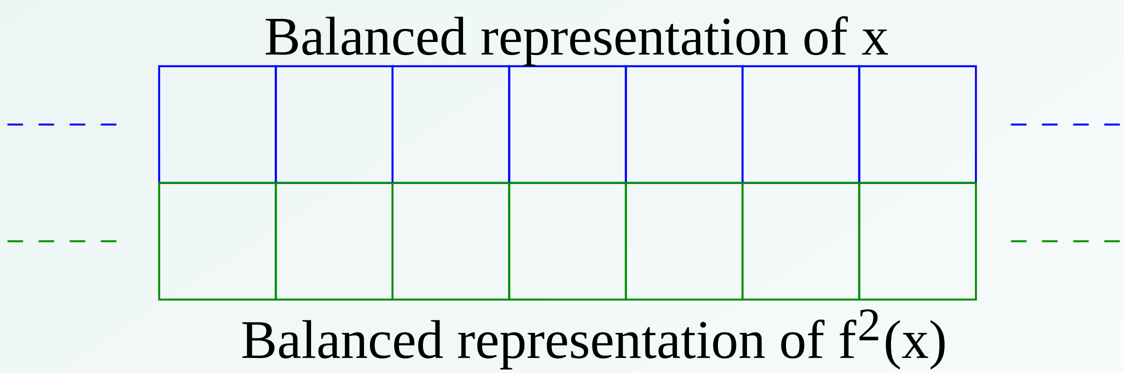

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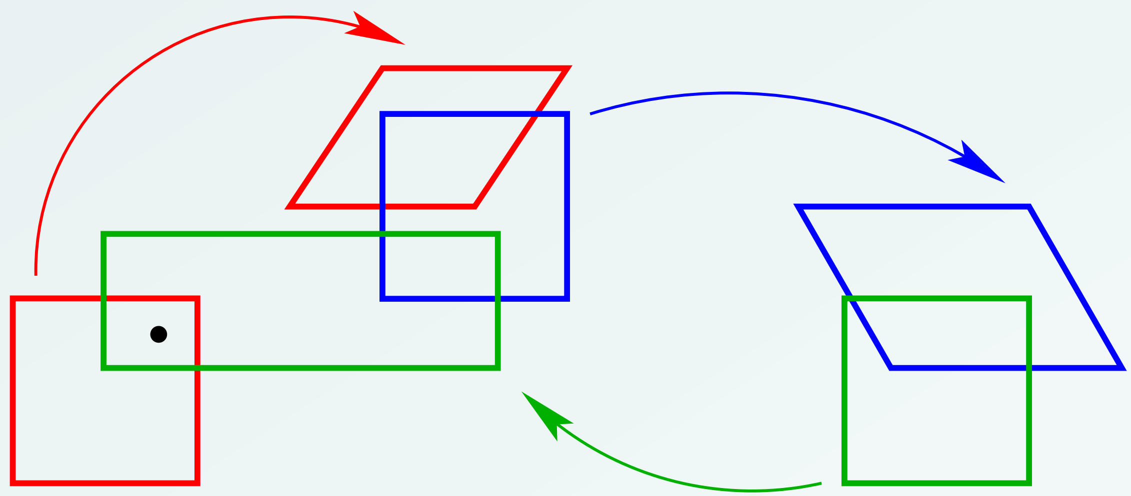
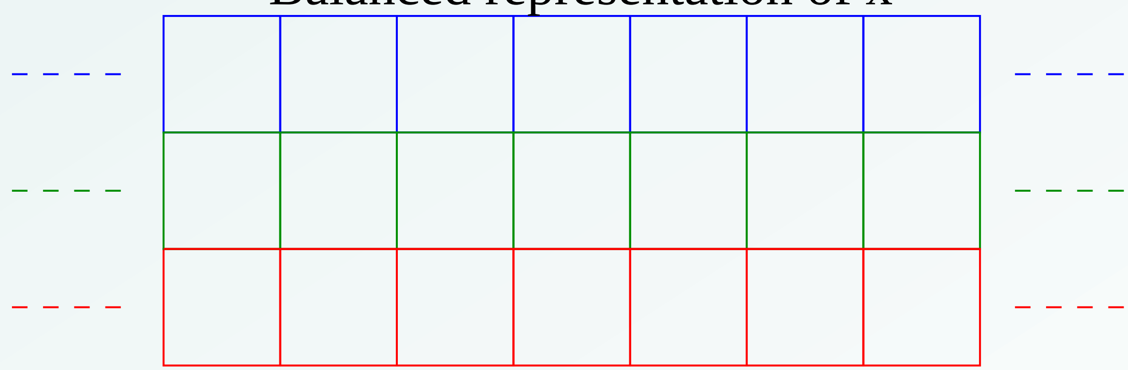
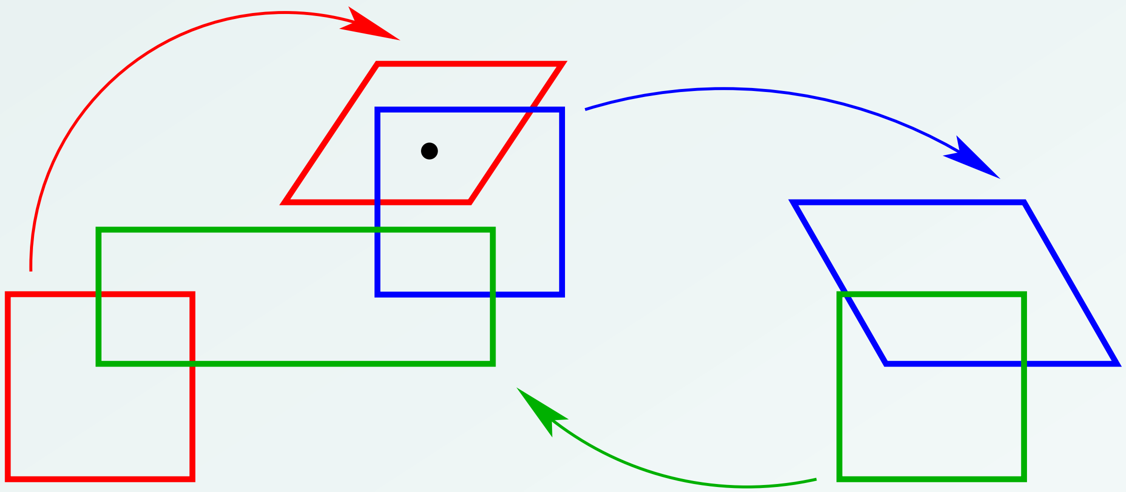


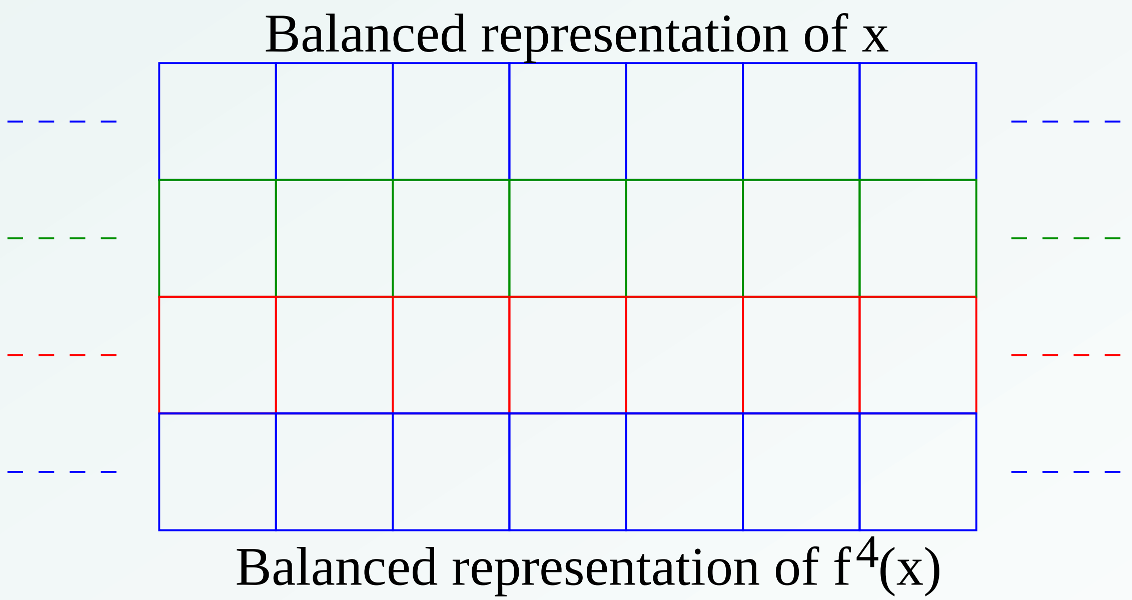
Balanced representation of x



Balanced representation of $f^3(x)$

If there is an infinite orbit then a tiling exists where the labels of the horizontal rows read the balanced representations of the points of the orbit:

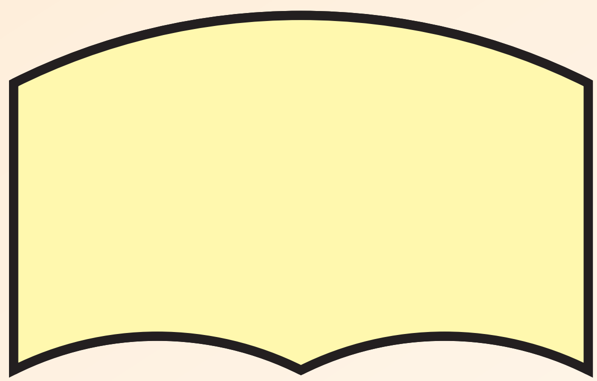




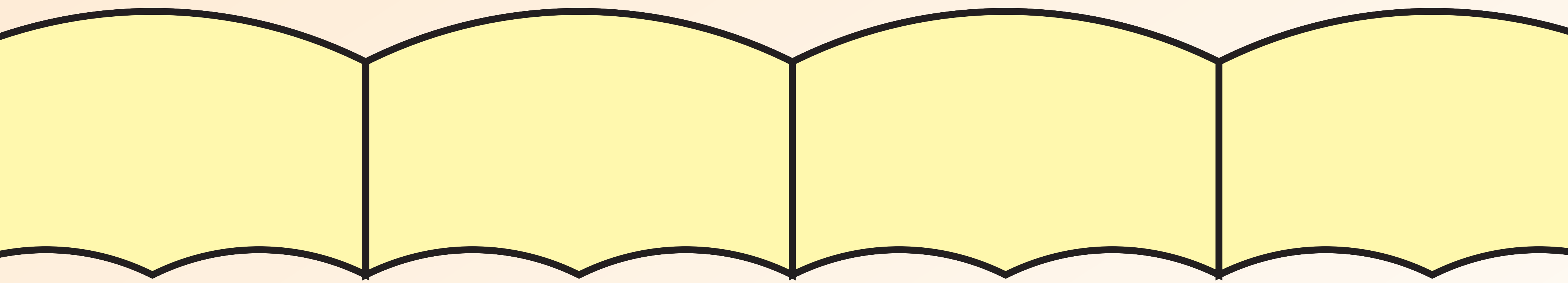
Conclusion: the tile set admits a tiling of the plane if and only if the system of affine maps is immortal. Undecidability of the tiling problem follows from the undecidability of the immortality problem. $\square$

The hyperbolic plane

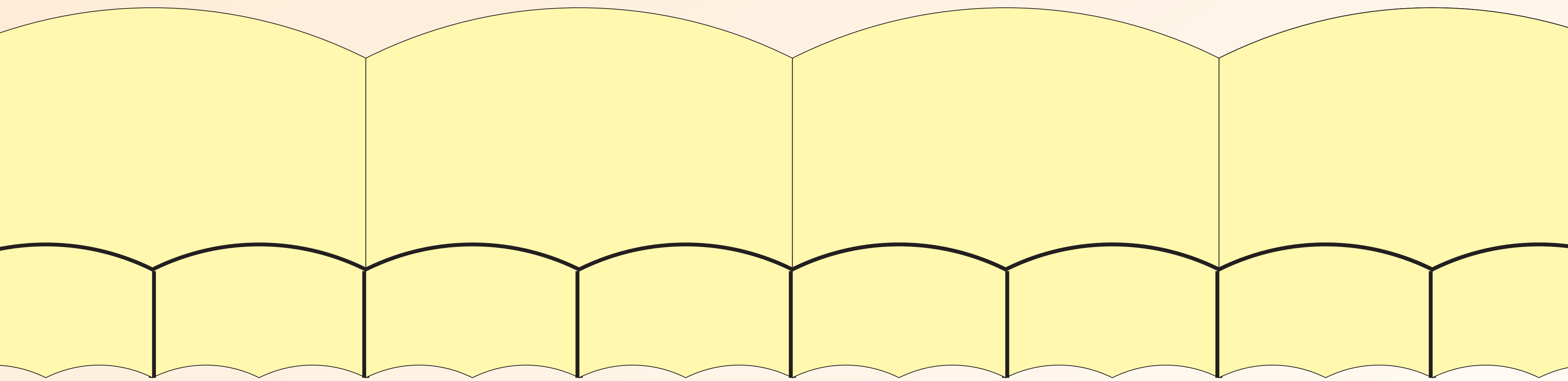
The technique works well also in the hyperbolic plane.



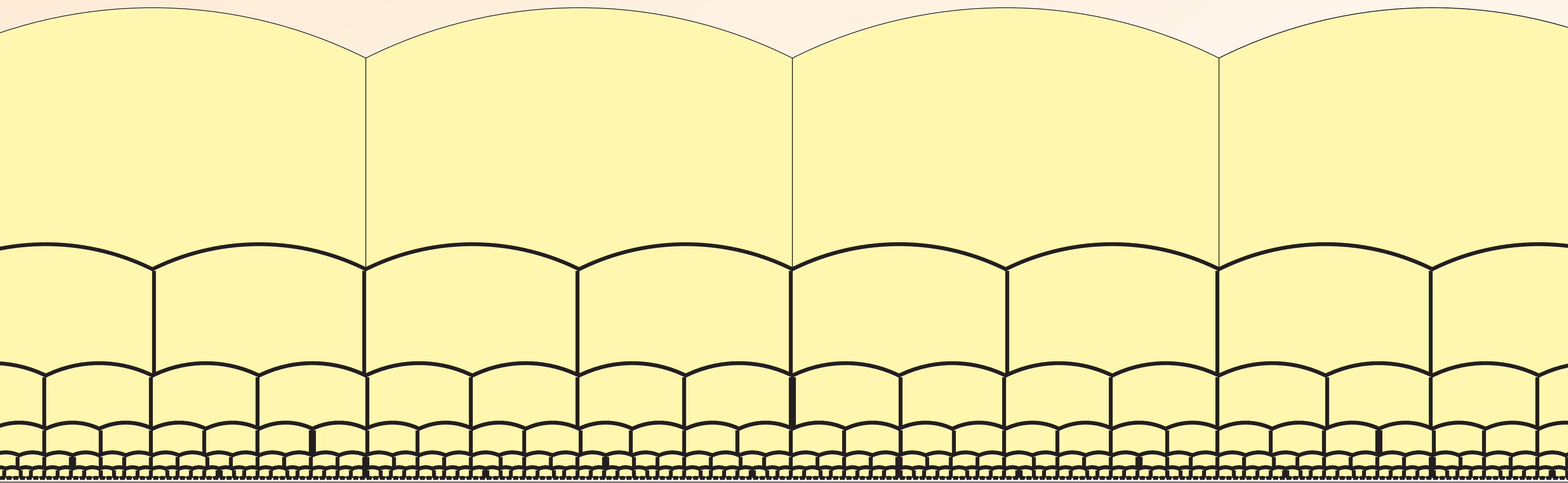
The role of the Euclidean Wang square tile will be played by a hyperbolic pentagon.



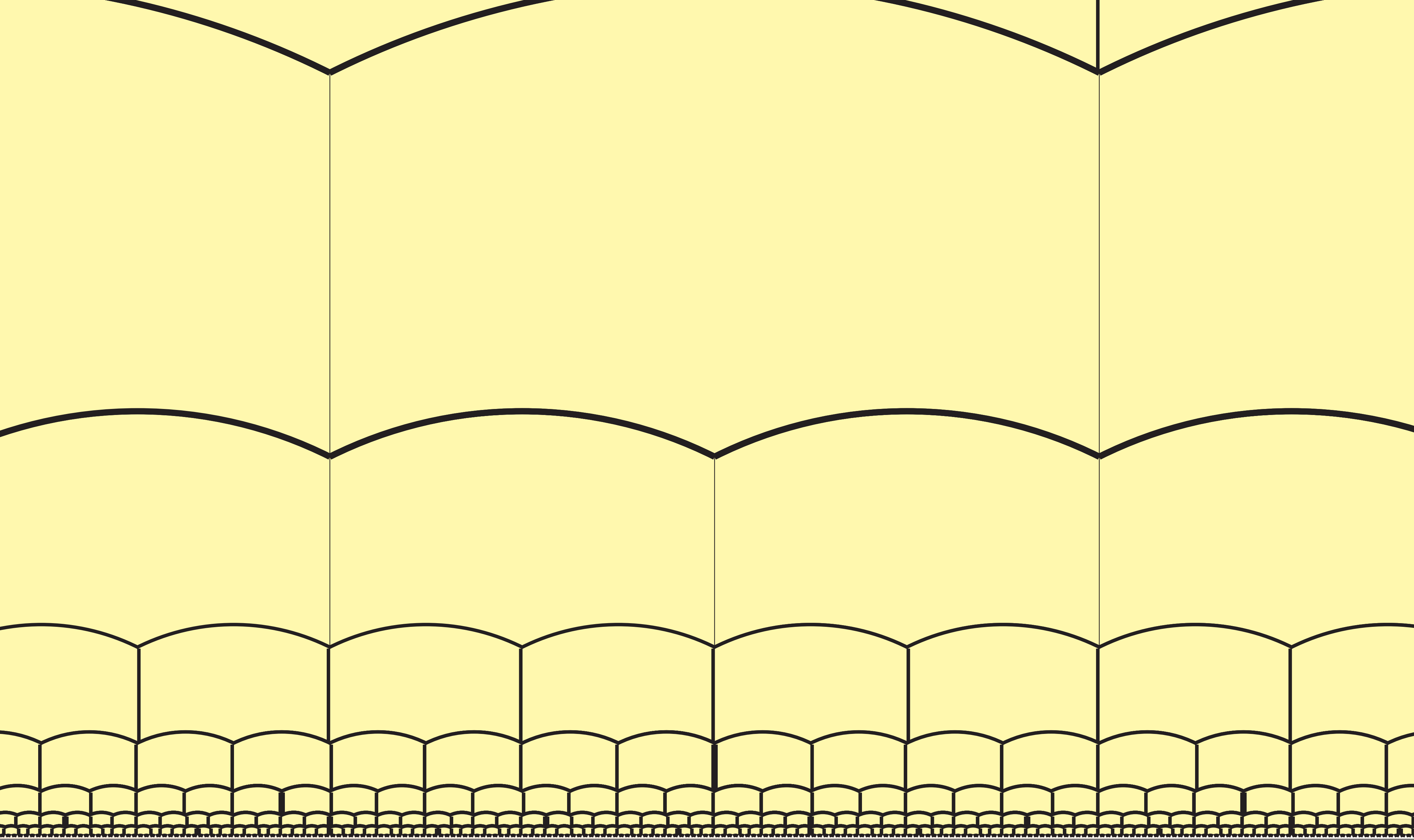
The pentagons can tile a "horizontal row".



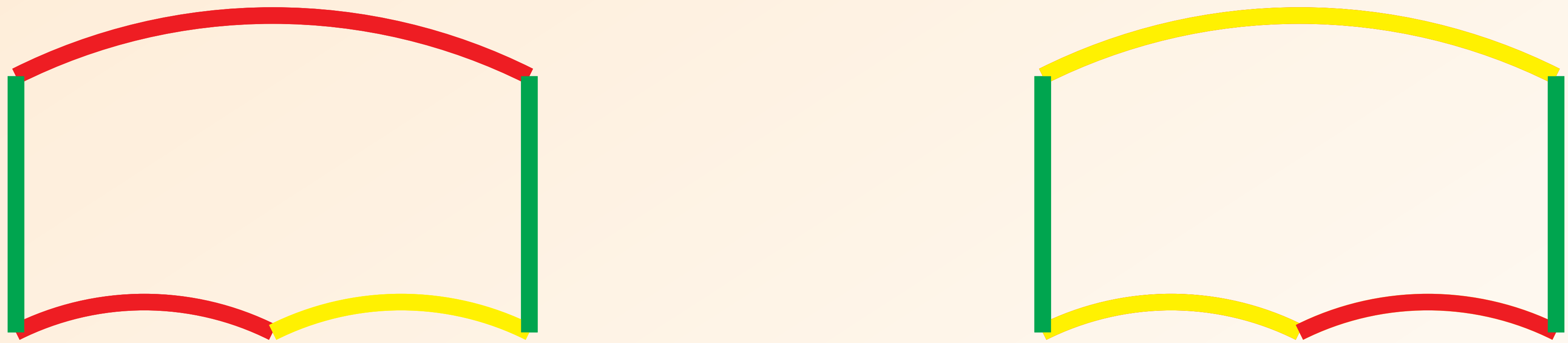
”Beneath” each pentagon fits two identical pentagons.



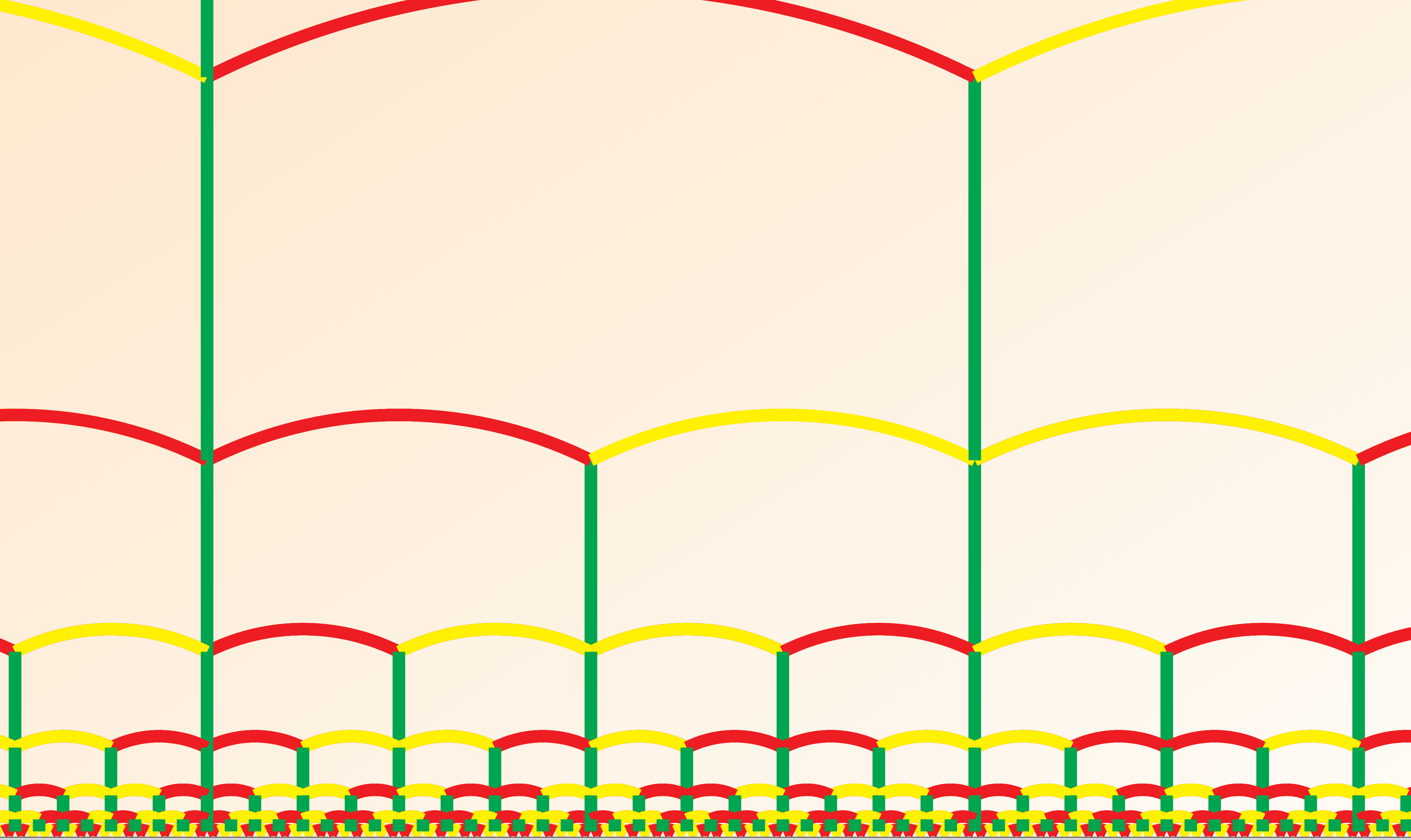
Infinitely many ”horizontal rows” fill the lower part of the half plane.



Similarily the upper part can be filled. We see that the pentagons tile the hyperbolic plane (in an uncountable number of different ways, in fact.)



On the hyperbolic plane Wang tiles are pentagons with colored edges. Pentagons may be placed adjacent if the edge colors match.

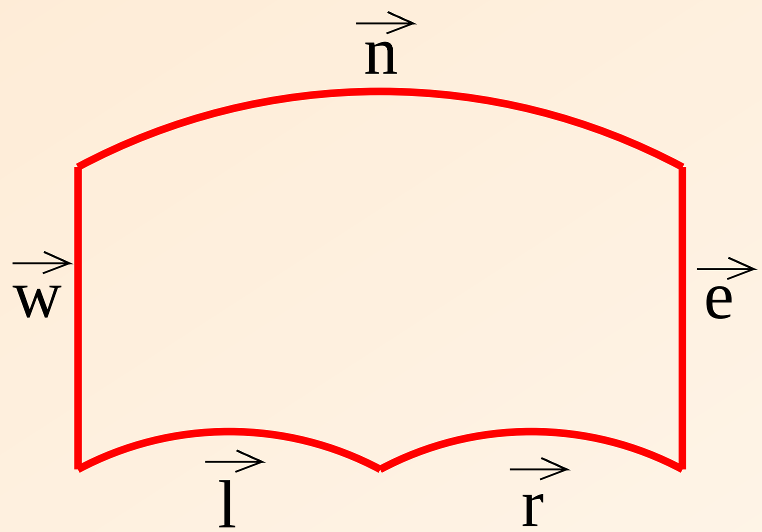


A given set of pentagons tiles the hyperbolic plane if a tiling exists where the color constraint is everywhere satisfied.

The hyperbolic tiling problem asks whether a given finite collection of colored pentagons admits a valid tiling.

Theorem. The tiling problem of the hyperbolic plane is undecidable.

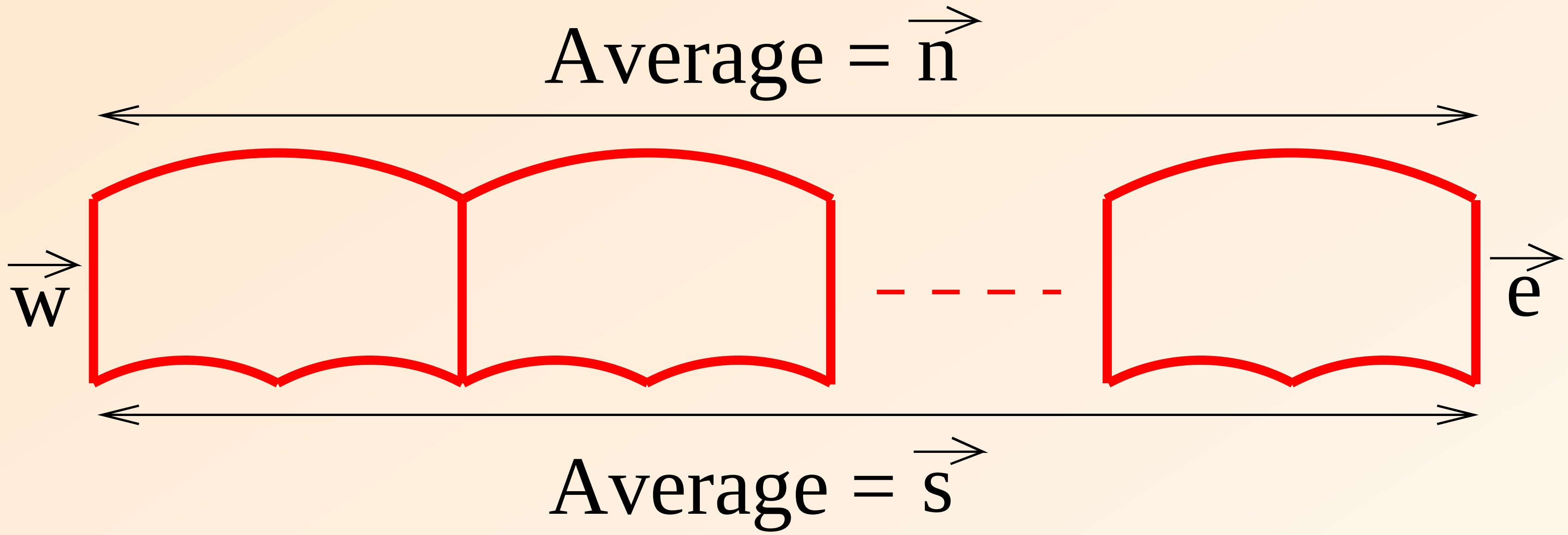
We say that pentagon



computes the affine transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ if

$$f(\vec{n}) + \vec{w} = \frac{\vec{l} + \vec{r}}{2} + \vec{e}.$$

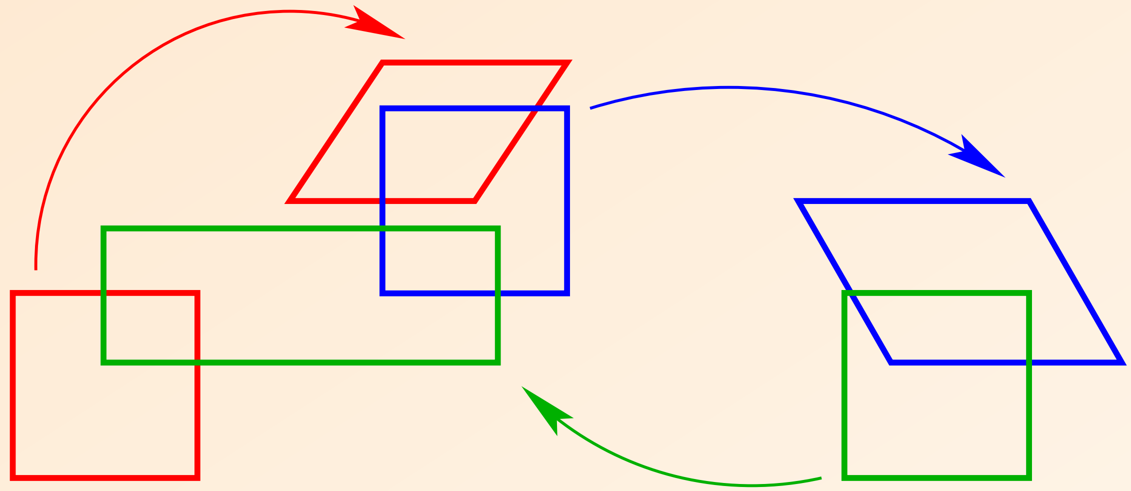
(Difference to Euclidean Wang tiles: The "output" is now divided between $\vec{l}$ and $\vec{r}$.)



In a horizontal segment of length n where all tiles compute the same f holds

$$f(\vec{n}) + \frac{1}{n}\vec{w} = \vec{s} + \frac{1}{n}\vec{e},$$

where $\vec{n}$ and $\vec{s}$ are the averages of the top and the bottom labels.

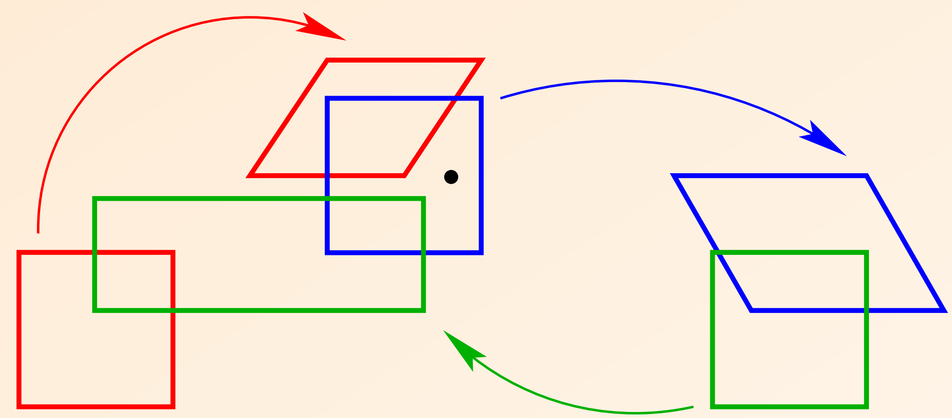


For a given system of affine maps f_i and unit squares U_i we construct for each i a set T_i of pentagons

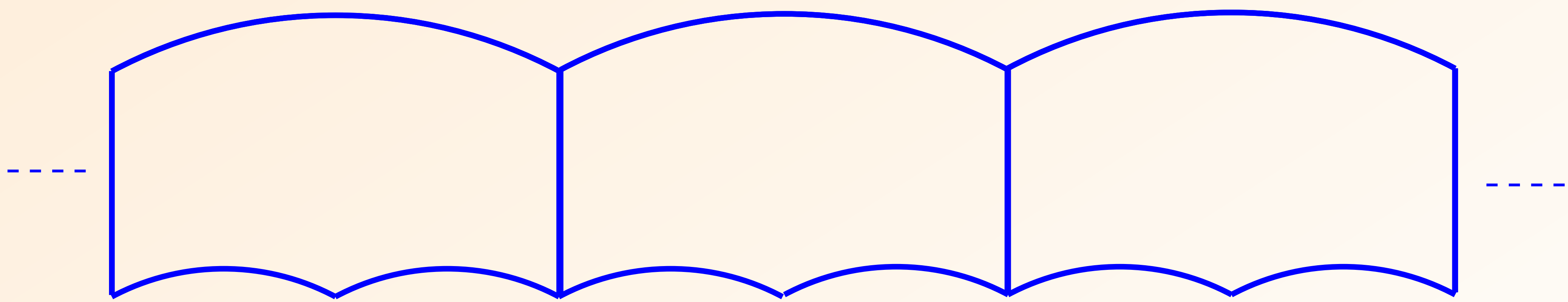
- that compute function f_i, and
- whose top edge labels $\vec{n}$ are in U_i.

It follows, exactly as in the Euclidean case, that valid tilings correspond to iterations of the piecewise affine maps.

The tiles constructed admit a valid tiling iff the system of affine maps has an immortal point:



Balanced representation of x



Balanced representation of f(x)

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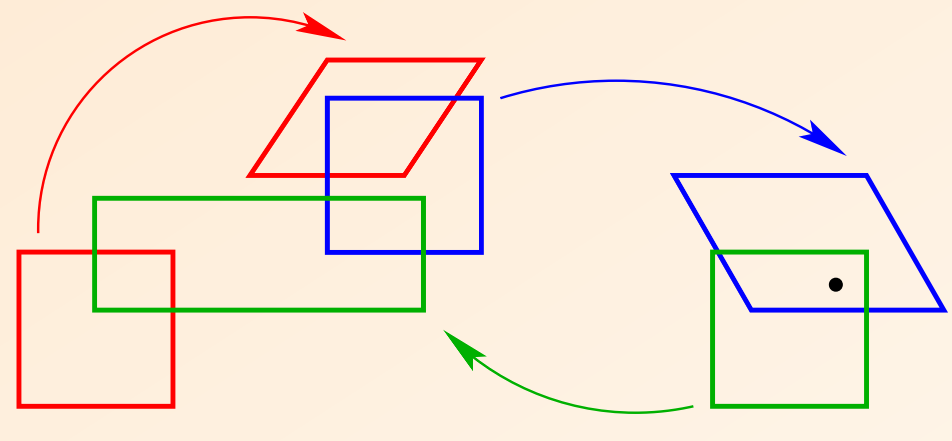
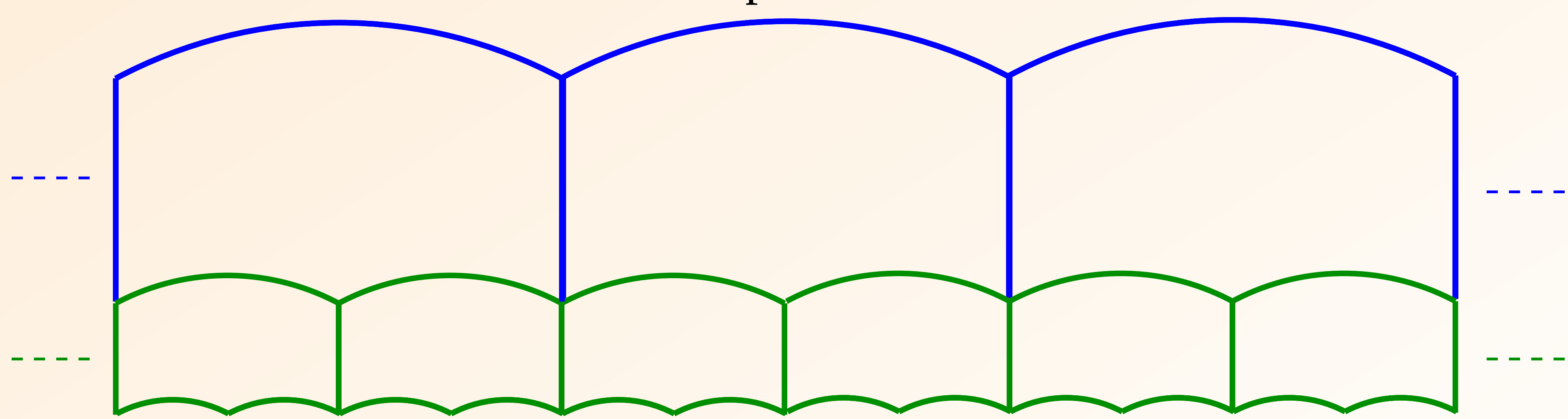
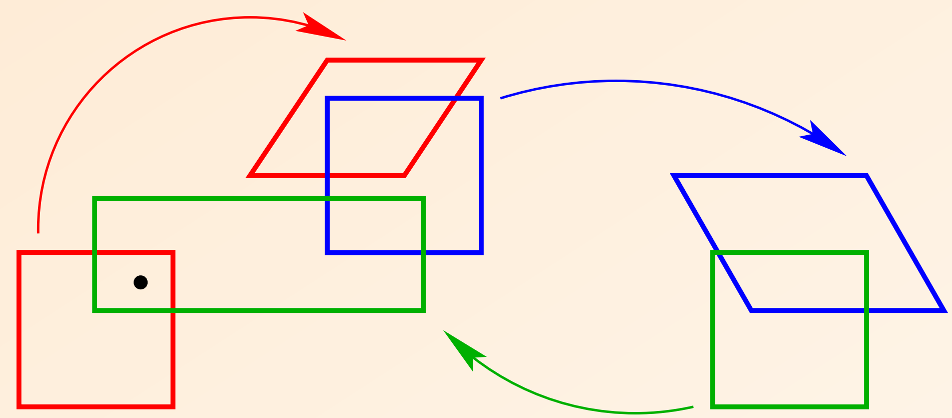


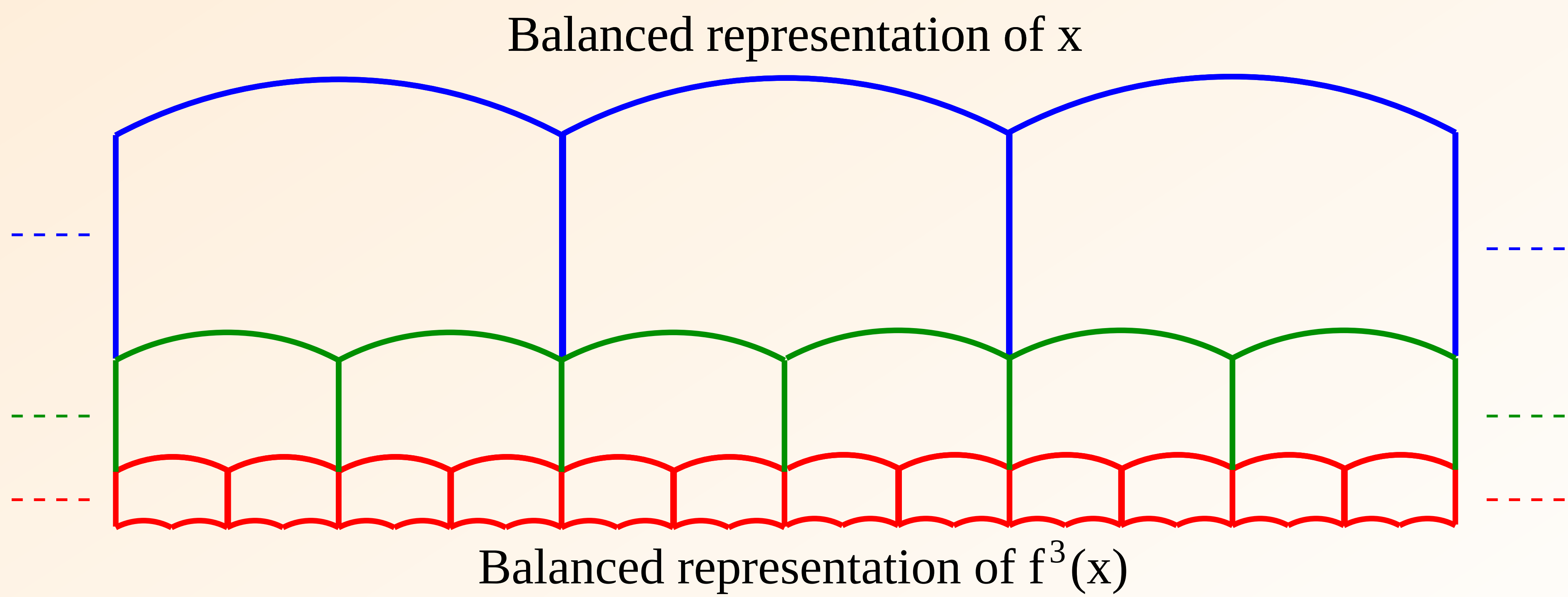
Balanced representation of x



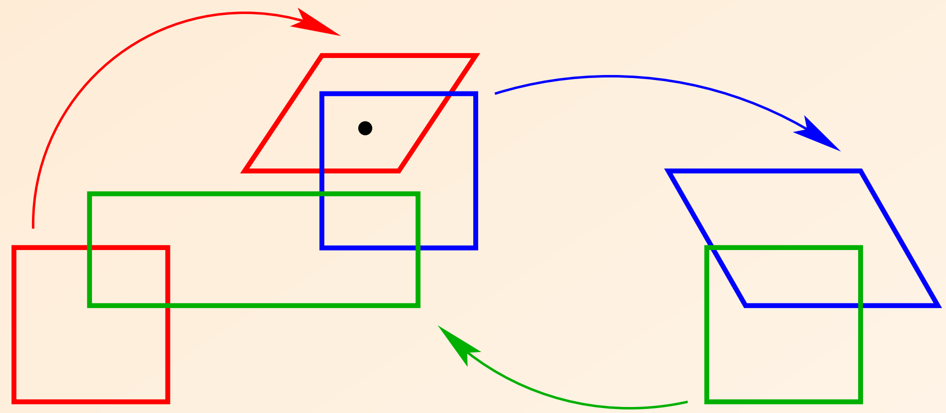
Balanced representation of $f^2(x)$

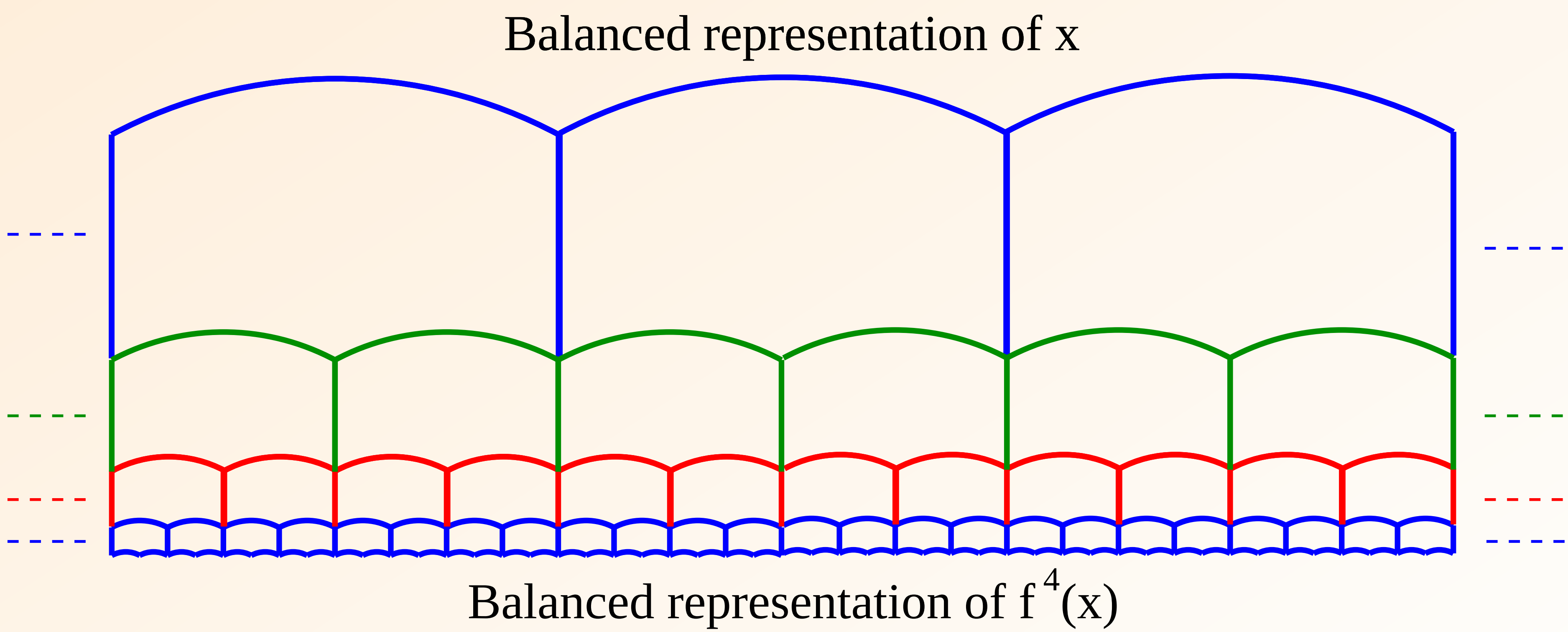
The tiles constructed admit a valid tiling iff the system of affine maps has an immortal point:





The tiles constructed admit a valid tiling iff the system of affine maps has an immortal point:





Conclusion

Sturmian representations of real numbers admit concise simulations of piecewise affine maps on 2D tilings.

$\Longrightarrow$ small aperiodic sets of Wang tiles

$\Longrightarrow$ simple undecidability proof of the tiling problem

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- Decidable on **virtually free** groups.
- Undecidable on **Baumslag-Solitar** groups (JK, N.Aubrun).