On vertices of degree k in minimal k-connected graphs

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Notations

For a graph G we use the following notations:

$V(G)$ denote the set of vertices;

$V_k(G)$ denote the set of vertices of degree k;

$v(G) = |V(G)|$ and $v_k(G) = |V_k(G)|$;

$E(G)$ denote the set of edges;

$d_G(x)$ denote the degree of a vertex x in the graph G;

$\Delta(G)$ denote the maximal vertex degree of the graph G.

k-connected graphs

definition

A graph is called *k-connected* if $v(G) \geq k+2$ and G remains connected after deleting any its k vertices.

definition

A k-connected graph is called *minimal*, if it becomes not k-connected after deleting any edge.

Clearly, all vertices of a k-connected graph have degree at least k.

Minimal biconnected graphs

G. A. Dirac (1967), M. D. Plummer (1968):

$$v_2(G) \geq \frac{v(G)+4}{3}$$

for a minimal biconnected graph G.

Minimal k-connected graphs

W. Mader (1979):

$$v_k(G) \geq \frac{(k-1)v(G)+2k}{2k-1} \tag{1}$$

for a minimal k-connected graph G.

This bound is tight. Clearly, the equality in (1) is possible only for $v(G) \equiv 2 \pmod{2k-1}$.

Extremal minimal k-connected graphs

definition

A minimal k-connected graph G is *extremal* if

$$v_k(G) = \left\lceil \frac{(k-1)v(G) + 2k}{2k-1} \right\rceil.$$

Denote by $\mathrm{GM}_k(n)$ the set of all extremal minimal k-connected graphs on n vertices.

definition

Let $f(G) = (2k-1)v_k(G) - (k-1)v(G) - 2k$ be the *defect* of a minimal k-connected graph G.

It follows from the Mader's inequlity that $f(G) \geq 0$. If G is extremal then $f(G) \leq 2k$. More precisely,

$$f(G) \equiv -(k-1)v(G) - 2k \pmod{2k-1}.$$

Graphs $G_{k,T}$

definition

Let $k \geq 2$ and T be a tree with $\Delta(T) \leq k+1$. The graph $G_{k,T}$ is constructed from k disjoint copies $T_1, \dots, T_k$ of the tree T. For any vertex $a \in V(T)$ we denote by a_i the correspondent vertex of the copy T_i. If $d_G(a) = j$ then we add $k+1-j$ new vertices of degree k that are adjacent to $\{a_1, \dots, a_k\}$.

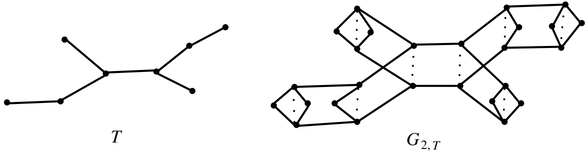


Figure: A tree T and correspondent extremal minimal biconnected graph $G_{2,T}$.

Characterization of $\mathrm{GM}((2k-1)t+2)$

The Mader's inequality turns to equality only for the graphs of the set $\mathrm{GM}_k((2k-1)t+2)$.
Clearly, if $v(T)=t$ then $v(G_{k,T})=(2k-1)t+2$. It is not difficult to verify that $G_{k,T}$ is an extremal minimal k-connected graph. That is $G_{k,T}\in \mathrm{GM}_k(2k-1)t+2))$.

Theorem

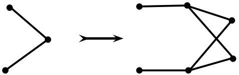
(DK, 01.2014).

The set $\mathrm{GM}_k(2k-1)t+2)$ consists of all graphs $G_{k,T}$ where T is a tree with $\Delta(T)\le k+1$ and $v(T)=t$.

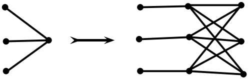
Algorithm of constructing extremal minimal graphs

J. G. Oxley (1982):

• any extremal minimal biconnected graph can be constructed by several opreations of substituting a vertex of degree 2 by $K_{2,2}$ (see fig. a) from some initial graph. The initial graphs are $K_{2,3}$ (for $\mathrm{GM}_2(3t+2)$), K_3, three graphs with more complicated structure and two infinite series of graphs;



a



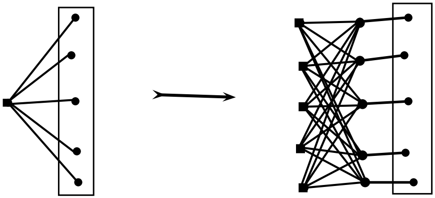
b

• any extremal minimal triconnected graph of the set $\mathrm{GM}_3(5t+2)$ can be constructed from $K_{3,4}$ by several opreations of substituting a vertex of degree 3 by $K_{3,3}$ (see fig. b).

Algorithm of constructing of extremal minimal graphs

Corollary

(DK, 2014)
Let $G \in \mathrm{GM}((2k-1)t+2)$. Then G can be constructed from $K_{k,k+1}$ by several opreations of substituting a vertex of degree k by $K_{k,k}$.



Note, that $K_{k,k+1}$ is isomorphic to the graph $G_{k,T}$ for a 1-vertex tree T.

More about $\mathrm{GM}(3t+1)$

Theorem

(DK,2013) Let $t \geq 2$. Then the following statements hold.

1) *The set $\mathrm{GM}(3t+1)$ consists of all graphs $G_{2,T} \cdot xy$, where T is a tree with $v(T) = t$ and $\Delta(T) \leq 3$, $x, y \in V_3(G_{2,T})$ and $xy \in E(G_T)$.*

2) *For any graph $G \in \mathrm{GM}(3k+1)$ the representation of type $G_T \cdot xy$ is unique up to isomorphism.*

More about $\mathrm{GM}(3t)$

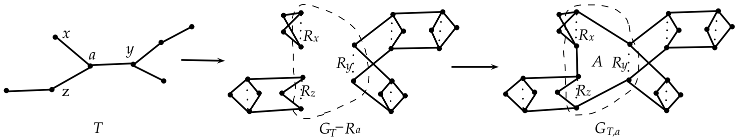
Theorem

(DK,2013) Let $t \geq 2$. Then the set $\mathrm{GM}(3t)$ consists of graphs of three types of graphs listed below:

1° graphs $G_T \cdot xy \cdot zt$, where T is a tree with $v(T) = t$ and $\Delta(T) \leq 3$, and $xy, zt \in E(G_T)$ are two distinct edges which ends have degree 3 in the graph G_T (these two edges may have a common end);

2° graphs, obtained from a graph $G_T - xy$ (where T is a tree with $v(T) = t - 1$ and $\Delta(T) \leq 3$, and $xy \in E(G_T)$) after adding a new vertex of degree 2, adjacent to x and y;

3° graphs $G_{T,a}$, where T is a tree with $v(T) = t$ and $\Delta(T) = 3$, and $a \in V(T)$ is a vertex of degree 3.



The case $k \geq 3$

The situation is quite different for $k \geq 3$.

Theorem

(W. Mader, 1979.) $\mathrm{GM}_k((2k-1)t+4) = \varnothing$ *for* $k \geq 3$ *and positive integer* t*.*

Conjecture

(W. Mader, 1979.) Let $v(G) = (2k-1)t + 2\ell$ *where* $2 \leq \ell \leq k-1$ *and* t *is a positive integer. Then*

$$v_k(G) \geq \left\lceil \frac{(k-1)v(G) + 2k}{2k-1} \right\rceil + 1.$$

Mader's conjecture

Mader's conjecture means that

$$\mathrm{GM}_k((2k-1)t+2\ell)=\varnothing$$

for $2\le\ell\le k-1$ and any positive integer t.

Mader has shown that $\mathrm{GM}_k((2k-1)t+p)\neq\varnothing$ for all other positive integer $p<2k-1$.

Why the case $v(G)=(2k-1)t+2\ell$ for $2\le\ell\le k-1$ differs so much from others?

A graph $G\in\mathrm{GM}_k((2k-1)t+2\ell)$ must have the defect $f(G)=\ell-1<k-1$. Thus the Mader's conjecture means that if $f(G)>0$ for a minimal k-connected graph G then $f(G)\ge k-1$.

A theorem on Mader's conjecture.

Theorem

(D. Karpov, 08.2014.) Let k, t, ℓ be positive integers such that $k \geq 3$ and $2 \leq \ell < \frac{4k+7+4\sqrt{k^2-k-2}}{9}$. Let G be a mininal k-connected graph with $v(G) = (2k-1)t + 2\ell$. Then

$$v_k(G) \geq \left\lceil \frac{(k-1)v(G) + 2k}{2k-1} \right\rceil + 1.$$

This bound is also best possible: there are several examples of graphs for which it is attained.

One can easily verify that $\frac{4k+7+4\sqrt{k^2-k-2}}{9} \geq \frac{8k+3}{9}$ for $k \geq 3$. That is we have proved Mader's conjecture for $\ell \leq \frac{8k+3}{9}$.

Extremal examples

Theorem

Let $t_k \geq k+1$ and $t_{k+1} \geq k$ be integers, such that

$$kt_k \geq (k+1)t_{k+1}.$$

Then there exists a bipartite minimal k-connected graph G with partitions $V_k(G)$ and $V_{k+1}(G)$ of sizes $|V_k(G)| = t_k$ and $|V_{k+1}(G)| = t_{k+1}$.

This theorem helps us to construct examples for which our bounds on vertices of degree k are tight for $v(G) = 2k - 1 + p$, where

- p is odd and $3 \leq p \leq 2k-1$;
- p is even and $2 \leq p \leq 2k-2$;

For $v(G) = (2k-1)t + p$ where $t \geq 1$ one can construct the extremal graph by t operations of substituting a vertex of degree k by $K_{k,k}$.

Extremal example for $v(G) = 2k - 1$.

$v_k = k + 1, \quad v_{k+1} = k - 2.$
The graph G is a union of complete bipartite graph with partitions V_k and V_{k+1} and a cycle C_{k+1} on the vertices of V_k.

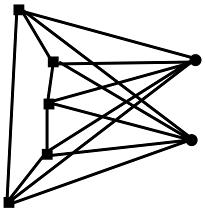


Figure: An example for $k = 4$, $v(G) = 7$.

Extremal examples for $v(G) = (2k-1)t+1$.

Let $t \geq 1$ and T be any tree with $\Delta(T) \leq k+1$ and $v(T) = T$, and x, y be adjacent vertices of the graph $G_{k,T}$. Then the graph $G_{k,T} \cdot xy \in \mathrm{GM}((2k-1)t+1)$.

It is easy to check that $\mathrm{GM}(2k) = \varnothing$, i.e. there is no minimal k-connected graph on $2k$ vertices with $v_k = k+1$ and $v_{k+1} = k-1$.

Conjecture

Any minimal k-connected graph of the set $\mathrm{GM}((2k-1)t+1)$ *is of type* $G_{k,T} \cdot xy$ *for some* $G_{k,T} \in \mathrm{GM}(((2k-1)t+2)$.

Thank You!