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On application of the probabilistic method to analysing the partitions of an integer

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Any representation of a positive integer in the form of a sum of positive integers (parts) is referred to as a partition of the number, which is primarily of combinatorial and number theoretical nature. Questions concerning the partitions have played an important part in mathematics.

Applications of the theory of partitions are found wherever discrete objects are to be counted or classified, whether in the molecular and the atomic studies of matter, in the theory of numbers, or in combinatorial problems from all sources.

Partitions are investigated in combinatorics and in the theory of numbers; classical combinatorial problems concern counting partitions, and in the theory of numbers, problems on additive representations of numbers are being solved under arithmetical constraints imposed on the parts.

Serious difficulties may arise while solving problems on partitions, though; so a great body of special methods in the theory of partitions have been elaborated. Historically, the first method which has since then become the most common in the whole theory of partitions is the method of generating functions. It was developed by Euler and has found application both in the theory of numbers and in combinatorics; it has been evolved into very delicate but universal tools.

If one considers the applications of partitions in various branches of mathematics, one is struck by the interplay of combinatorial and asymptotic methods.

In the application of partitions (such as in statistics) we are often interested in restricted partitions.

So, we consider the problem to count the partitions of a positive integer n into exactly s positive integer parts which do not exceed a given integer r (those partitions which differ in the order of parts only are counted as one).

$$n = k_1 + k_2 + \cdots + k_s,$$

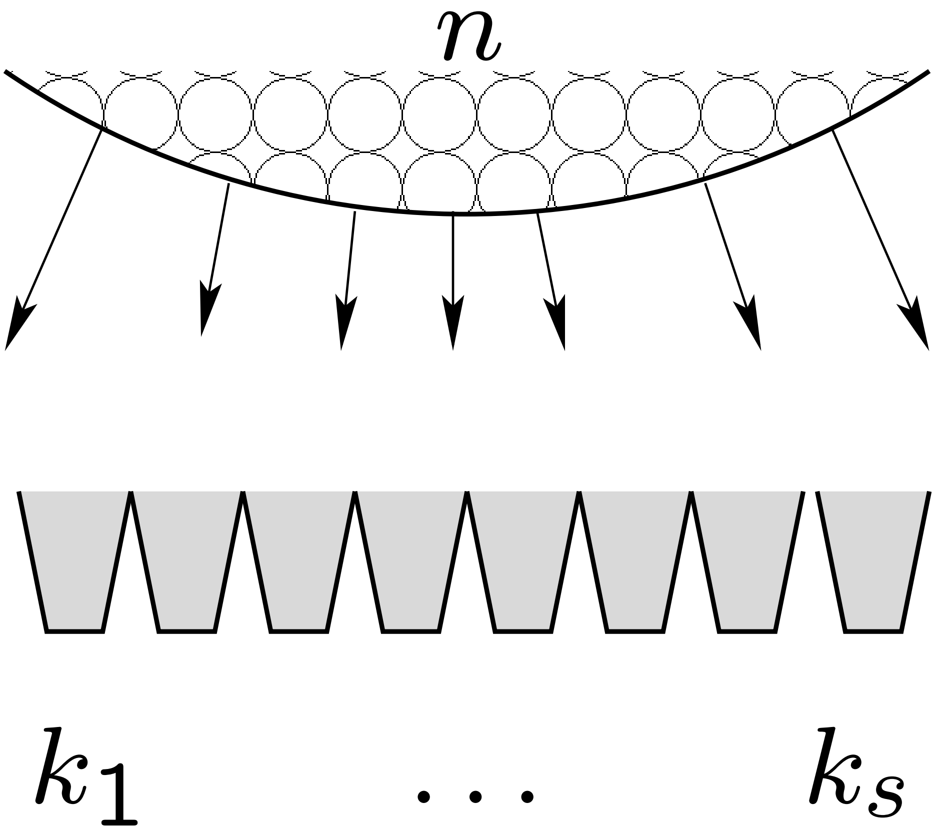
$k_1, k_2, \ldots, k_s \le r.$

Let $C_{n,s,r}$ stand for the number of these partitions.

It is quite easy to find the generating function for this number, which could lead us to analysis of the Gaussian polynomials.

But we use the *The Probabilistic Method:*

$\Rightarrow$ Classical Occupancy Problem



where $k_1, \ldots, k_s \leq r$.

The partition of a number is thus described by the classical scheme of equiprobable allocation of n indistinguishable particles into s indistinguishable cells under the condition that each cell holds from one to no more than r particles.

Let $\xi_1, \xi_2, \ldots, \xi_s$ be independent random variables which take values $1, 2, \ldots, r$ with equal probabilities, which can be considered as the contents of the corresponding cells in the above allocation scheme. It is easily seen that

$$\mathbf{P}\{\xi_1 + \xi_2 + \cdots + \xi_s = n\}$$
$$= \sum_{\substack{k_1+\cdots+k_s=n \\ k_1,\ldots,k_s \le r}} \mathbf{P}\{\xi_1 = k_1, \ldots, \xi_s = k_s\} = C_{n,s,r} \frac{1}{r^s}; \quad (*)$$

In addition, the mean and the variance obey the equalities

$$\mathbf{E}\xi_1 = \frac{r+1}{2} = m,$$
$$\mathbf{Var}\,\xi_1 = \frac{r^2-1}{12} = \sigma^2.$$

So, in order to investigate the asymptotic behaviour of $C_{n,s,r}$ one is able to utilise the well-developed apparatus of *local limit theorems of probability theory*.

In the ‘central’ domain, the normal approximation holds.

In other domains of variation of the parameters n, s, r, convergence to various stable laws occurs.

It is clear that

$$\mathbf{P}\{\xi_1+\xi_2+\cdots+\xi_s=n\}$$
$$=\mathbf{P}\left\{\frac{\xi_1+\cdots+\xi_s-sm}{\sigma}=\frac{n-sm}{\sigma}\right\}.$$

Let

$$x=\frac{n-sm}{\sigma}.$$

Using the expressions of m and σ, we obtain

$$x=\frac{2n-s(r+1)}{\sqrt{(r^2-1)/3}}.$$

The *Central* domain of variation of the parameters.

For the sake of simplicity, let r be fixed. From relation (*), with the use of the local convergence to the standard normal law we see that for fixed r, while the parameters n and s tend to infinity in such a way that the ratio of n to s lies in some finite interval, the relation

$$\frac{1}{r^s} C_{n,s,r} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}(1 + o(1))$$

holds true uniformly with respect to all x in an arbitrary fixed finite interval, hence the relation

$$C_{n,s,r} = \frac{r^s}{\sqrt{2\pi}} e^{-x^2/2}(1 + o(1)),$$

is true uniformly with respect to those x; we recall that

$$x = \frac{2n - s(r + 1)}{\sqrt{(r^2 - 1)/3}}.$$

Of interest is also the

Inverse Problem:

to study asymptotic properties of characteristics of a classical scheme of allocation of particles to cells with the use of apparatus of the theory of partitions and compositions of an integer.

Thank you for your attention!