Minimum number of input clues in an associative memory

Ville Junnila and Tero Laihonen

Department of Mathematics and Statistics, University of Turku, Finland

Introduction

- Yaakobi and Bruck (2012): How to retrieve information from associative memories
- An associative memory is modeled by a graph.
- Let $G = (V, E)$ be a simple, undirected and connected graph
- $d(u, v)$ the graphic distance
- the ball of radius t

$$B_t(x) = \{v \in V \mid d(x, v) \leq t\}.$$

Introduction

- Information units are stored in vertices
- an edge means associations between information units
- We say that $u \in V$ and $v \in V$ are t-**associated** if $d(u, v) \leq t$.
- $B_t(x)$ is the set of vertices t-associated to x.
- A reference set $C \subseteq V$. It is nonempty and we call it a *code* and its elements *codewords*.

Retrieval of information unit

- Suppose we wish to retrieve an (unknown) information unit $x \in V$.
- We receive **input clues** from C, which are t-associated to x.
- In other words, input clues come from

$$I_t(x) = B_t(x) \cap C.$$

- Input clues come one after another

Retrieval of information unit

- After receiving a new input clue, we check which vertices are t-associated to all input clues so far
- Suppose that $U \subseteq I_t(x)$ has been received. We calculate an output set

$$S_t(U) = \bigcap_{c \in U} B_t(c).$$

- Clearly, $x \in S_t(U)$
- It is convenient to define $S_t(\emptyset) = V$.

Limit on uncertainty

- We set a limit $N \geq 1$ called **uncertainty**.
- We want to know x with (small) uncertainty

$$|S_t(U)| \leq N$$

or uniquely

$$S_t(U) = \{x\}.$$

- Number of input clues needed?

Number of input clues

- For each $x \in V$ we define a function $m_t^N(x) = m_t^N(C;x)$.
- We set

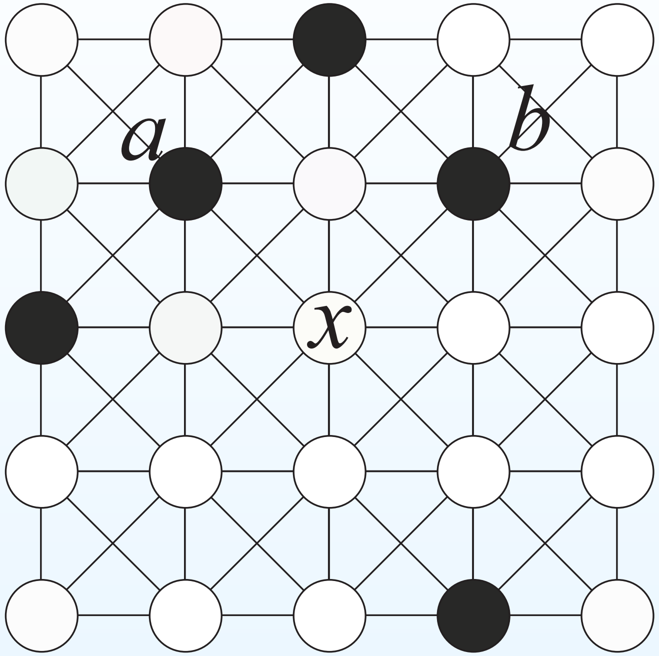
$$m_t^N(x) = \infty$$

  if $|S_t(I_t(x))| > N$
- in this case, even the full set of input clues is not enough to meet the desired uncertainty
- If $|S_t(I_t(x))| \leq N$, then we define

$$m_t^N(x) = s$$

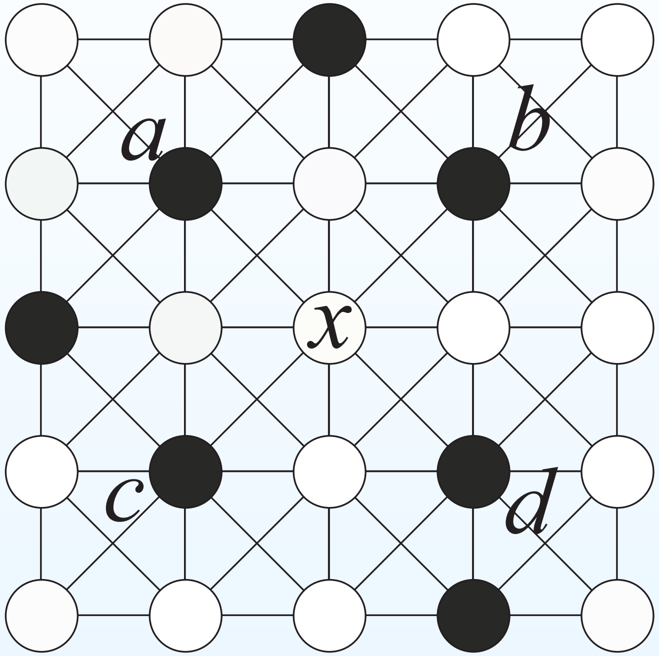
  where s is the minimum integer such that for **any** $U \subseteq I_t(x)$ with $|U| = s$ we have $|S_t(U)| \leq N$.

An example



- The case $N = 1$ and $t = 1$. Now $m_1^1(x) = \infty$, because $|S_1(I_1(x))| = 3$.

An example



- It is always enough to listen to at most s input clues
- $m_1^1(x) = 3.$

Definition

- Naturally we want to find every $x \in V$ with given uncertainty N
- A code C gives an $\mathcal{SAM}_G(t;N)$ if

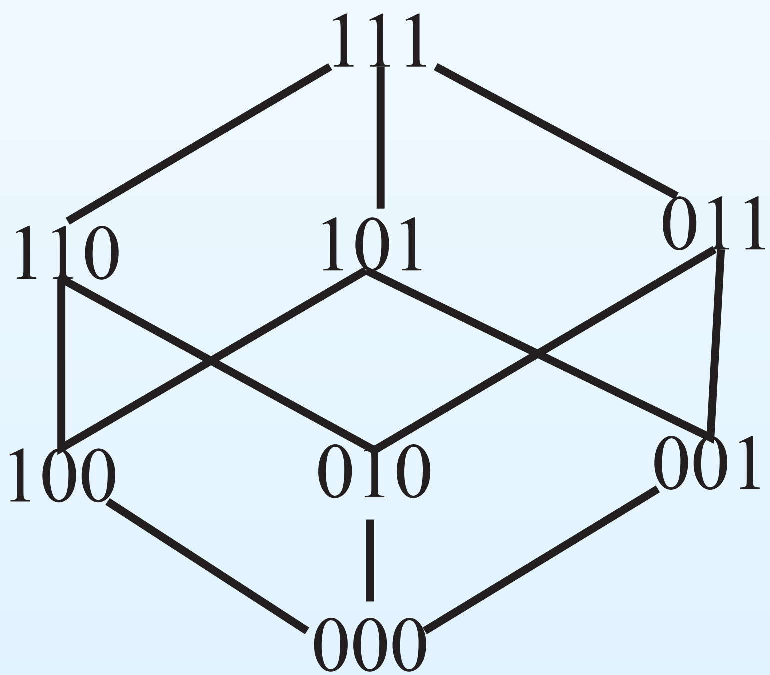
$$m_t^N(x) < \infty$$

  for all $x \in V$.
- Given t and N, we optimize
  - upper bound m_u where $m_t^N(x) \leq m_u$ for all x
  - $m_{av} = \frac{1}{|V|} \sum_{x \in V} m_t^N(x)$
  - fixed m with $m_t^N(x) = m$ for all x
- Given m and t find minimum N

- Codes giving an $\mathcal{SAM}_G(t;N)$ have been studied in
  - binary Hamming spaces $\mathbb{F}^n$
  - infinite square grid
  - infinite king grid
  - Grassmann graphs
- General (undirected) graphs for $N=1$ and $t=1$.

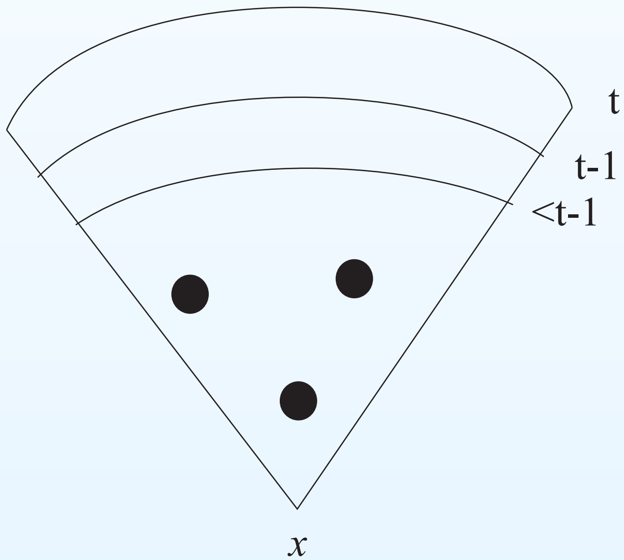
Hamming space and $N = 1$

- The binary Hamming space:
  - $\mathbb{F} = \{0, 1\}$.
  - a vertex (a word) $x_1 x_2 \ldots x_n \in \mathbb{F}^n$
  - An edge between x and y if they differ in exactly one coordinate
- $\mathbb{F}^3$



Structure for $N = 1$ and $t \geq 2$

- Let C give an $\mathcal{SAM}(t, 1)$. If $U \subseteq I_t(x)$ and $|U| \geq m_t(x)$, then there are $c_1, c_2, c_3 \in U$ such that $d(c_1, x) = d(c_2, x) = t$ and $t - 1 \leq d(c_3, x) \leq t$.



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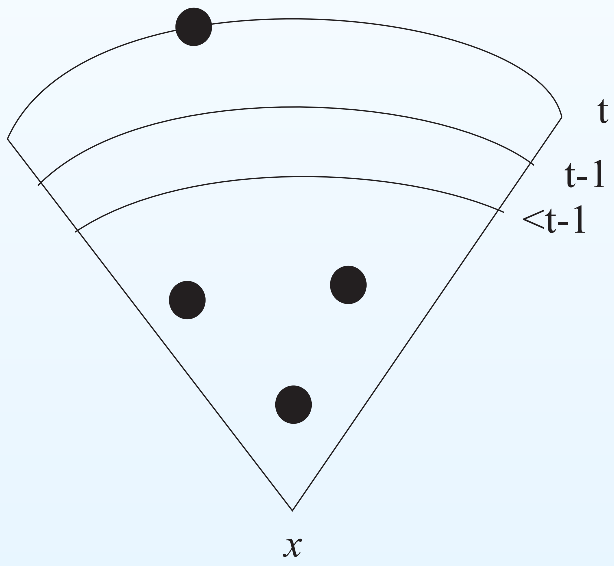

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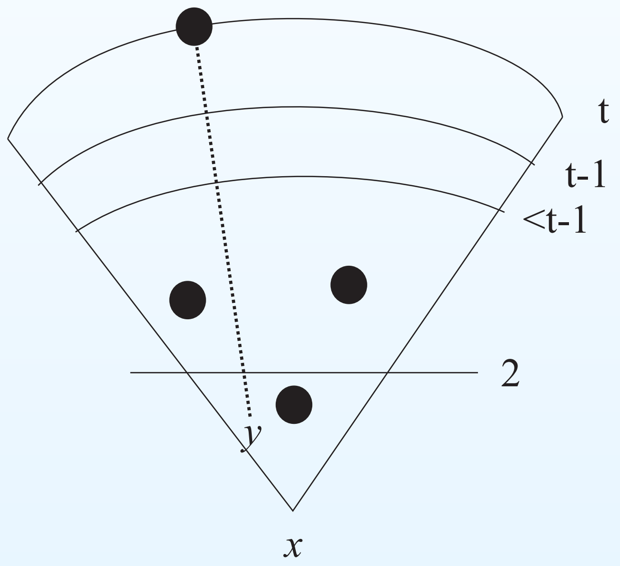
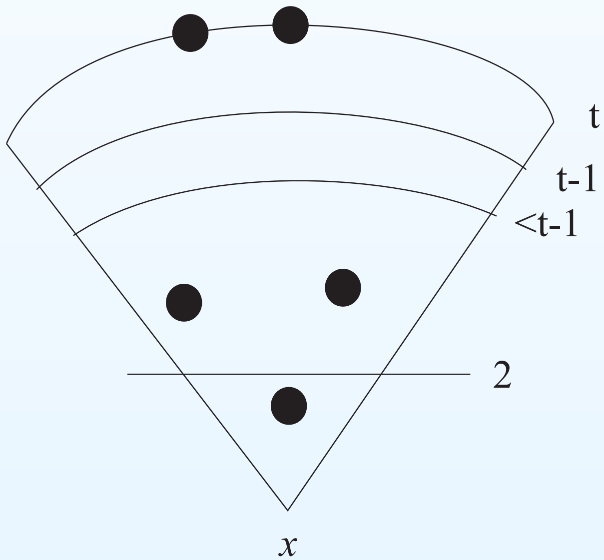


Let $x = 000\ldots$.
If $\mathbf{c} = 00111\ldots0001$, then $y = 001000\ldots$.
Now $|S_t(I_t(x))| \geq 2$.

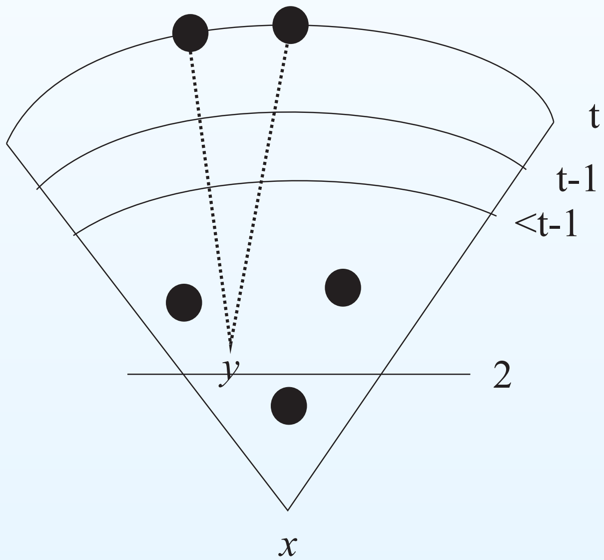
Structure for $N = 1$ and $t \geq 2$

- Let C give an $\mathcal{SAM}(t, 1)$. If $U \subseteq I_t(x)$ and $|U| \geq m_t(x)$, then there are $c_1, c_2, c_3 \in U$ such that $d(c_1, x) = d(c_2, x) = t$ and $t - 1 \leq d(c_3, x) \leq t$.



Structure for $N = 1$ and $t \geq 2$

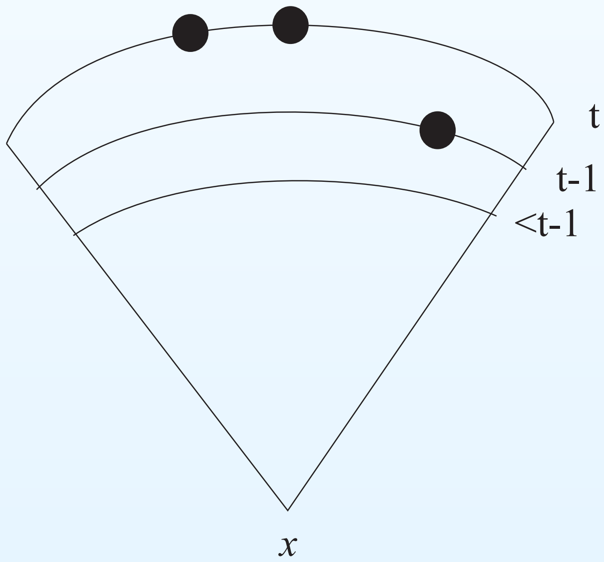
- Let C give an $\mathcal{SAM}(t,1)$. If $U \subseteq I_t(x)$ and $|U| \geq m_t(x)$, then there are $c_1, c_2, c_3 \in U$ such that $d(c_1,x) = d(c_2,x) = t$ and $t-1 \leq d(c_3,x) \leq t$.



If $\mathbf{c}_1 = 100100\ldots$ and $\mathbf{c}_2 = 0000011100\ldots$ choose $y = 10000100\ldots$

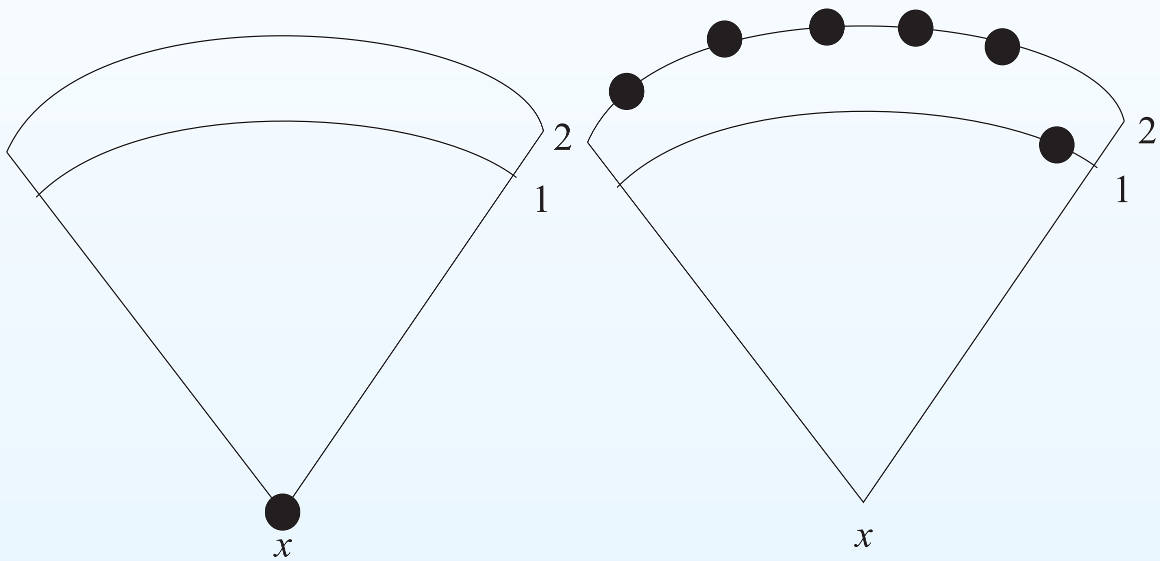
Structure for $N = 1$ and $t \geq 2$

- Let C give an $\mathcal{SAM}(t, 1)$. If $U \subseteq I_t(x)$ and $|U| \geq m_t(x)$, then there are $c_1, c_2, c_3 \in U$ such that $d(c_1, x) = d(c_2, x) = t$ and $t - 1 \leq d(c_3, x) \leq t$.
- $m_u \geq 4$ if $t \geq 2$ and $m_u \geq 5$ for $t \geq 4$.



$N = 1$ and $t = 2$: Constructing codes

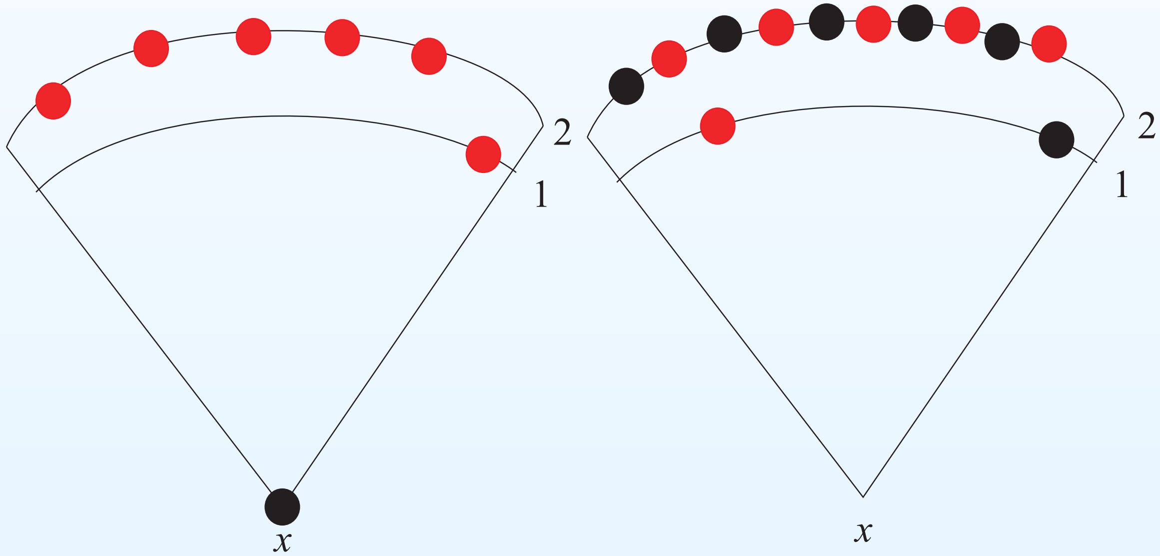
- For $t = 2$ we can utilize the Hamming codes $\mathcal{H}_r$ of length $n = 2^r - 1$:



Distance ≥ 3. No $\mathbf{c}_1 = 11000\ldots$ and $\mathbf{c}_2 = 10001\ldots$
For such codewords, three intersect uniquely in x.

$N = 1$ and $t = 2$: Constructing codes

- For $t = 2$ we can utilize the Hamming codes $\mathcal{H}_r$ of length $n = 2^r - 1$:
- $C = \mathcal{H}_r \cup (10000\ldots0 + \mathcal{H}_r)$

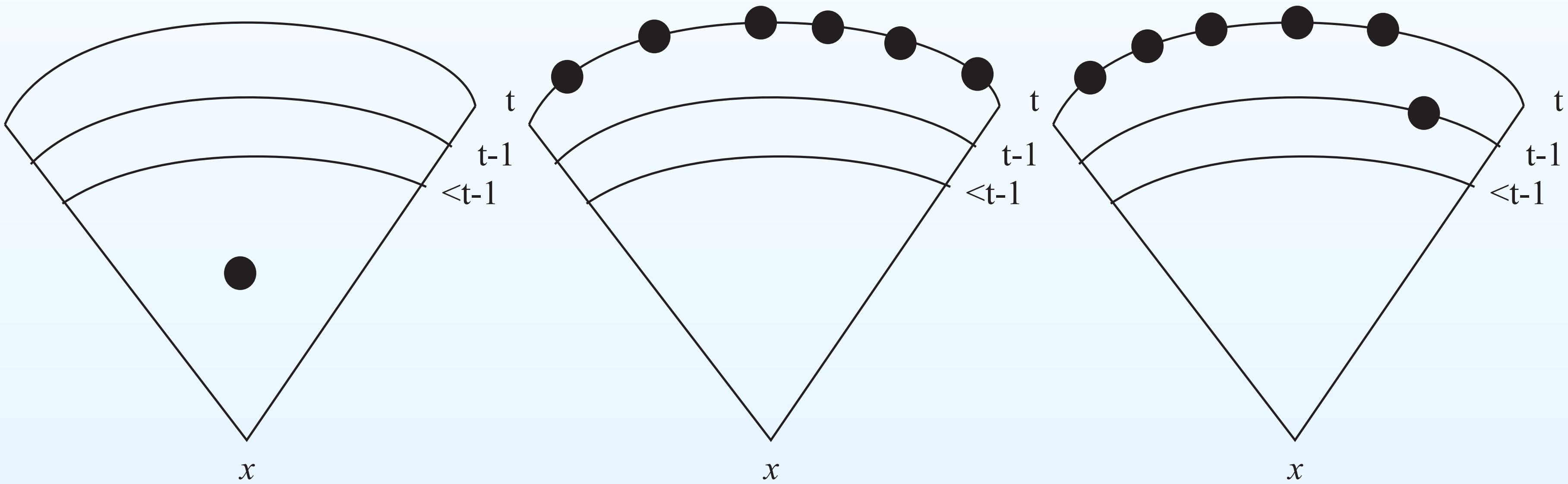


$N = 1$ and $t = 2$: Constructing codes

- For $t = 2$ we can utilize the Hamming codes $\mathcal{H}_r$ of length $n = 2^r - 1$:
- $C = \mathcal{H}_r \cup (10000\ldots 0 + \mathcal{H}_r)$
- Gives $m_u \leq 5$.
- For linear codes $m_u = 5$ is optimal and we can get it for each n
- For $t = 1$ we have $m_u = 3$ optimal.

$N = 1$ and $t = 3$:Constructing codes

- Let $t = 3$. For the punctured Preparata code $\mathcal{P}_r$ of length $n = 2^{2r} - 1$, $r \geq 2$ we have



Now the distance ≥ 5.

$N = 1$ and $t = 3$:Constructing codes

- Let $t = 3$. For the punctured Preparata code $\mathcal{P}_r$ of length $n = 2^{2r} - 1$, $r \geq 2$ we have
- $C = \mathcal{P}_r \cup (11000\ldots0 + \mathcal{P}_r) \cup (00110\ldots0 + \mathcal{P}_r)$
- Gives $m_u \leq 7$.
- We can use also primitive two-error correcting BCH codes of length $n = 2^{2r+1} - 1$, $r \geq 2$
- Shortening method gives other lengths.
- A code giving $\mathcal{SAM}(t;1)$ gives also $\mathcal{SAM}(n-t-1;1)$.

Undirected graph G: Fixed m

- If G admits an $\mathcal{SAM}(1,1)$, then

$$3 \leq m \leq \delta + 1.$$

- These can be attained:
  - Any 3-fold covering in a graph with girth ≥ 5.
  - The complete bipartite graph $K_{s,r}$ admits an $\mathcal{SAM}(1,1)$ if $s = r$ and $m = s + 1$, $s \geq 2$.
- We have

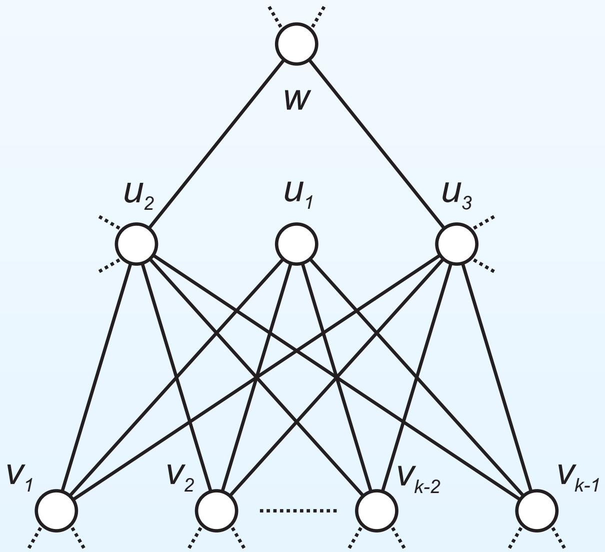
$$m \geq \frac{\triangle(\triangle + 1)}{\triangle(\triangle + 1) - \delta\Omega}$$

where $\Omega = \min_{x \sim y} |B_1(x) \cap B_1(y)|$.

- This can be attained: K_n minus a perfect matching

Fixed m and Forced vertices

- A vertex is a **forced codeword** if it belongs to all reference set giving $\mathcal{SAM}(1,1)$.
- A vertex is a **forced non-codeword** if it does not belong to all reference set giving $\mathcal{SAM}(1,1)$.



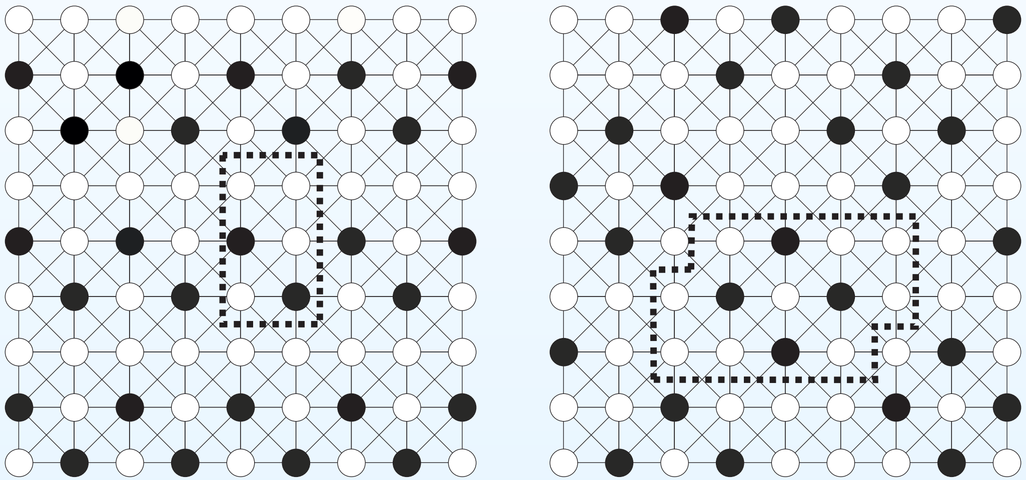
Here w is forced non-codeword, because of u_1, u_2 and u_3.

Forced vertices

- Let $|C| = K$. How many vertices can be forced non-codewords in a graph?
- There exist graphs with $\binom{K}{m} - K$ forced non-codewords and K forced codewords for any $m \geq 3$ and $K \geq m + 2$. This is the maximum also.

Average m_{av}

- In the infinite king grid we have:
  - optimal $m_{av} = 35/13$ for $N = 2$ and $t = 1$.
  - optimal $m_{av} = 8/3$ for $N = 3$ and $t = 1$



(a) (b)

For general t we have

$$2t/3 \lesssim m_t^3(x) \lesssim 2t - \sqrt{2t}.$$

Thank you!