Undecidability for integer weighted Büchi automata and Robot Games with states

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- Similarly defined when $w \in A^\omega$.

- For given Finite Automaton $\mathcal{A}$, over alphabet A, is its language $L(\mathcal{A}) = A^*$?
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- Both are known to be decidable.

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- For given Weighted Automaton $\mathcal{A}^{\gamma}$, over alphabet *A*, is its language $L(\mathcal{A}^{\gamma}) = A^*$?
- Shown to be undecidable by Halava and Harju in 1999.

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- Let $\gamma : T \to \mathbb{Z}$ be the weight function that assigns weights to edges.
- Let w be an infinite word. It is accepted by $\mathcal{A}^\gamma$ if there exists a computation π such that infinite number of its prefixes p have zero weight.

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- Shown to be undecidable by Halava and Harju for domain alphabets $|A| \geq 9$ and improved to $|A| \geq 8$ by Dong and Liu.

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- The solutions are of specific form:
  1. The first letter is d that contains the input of semi-Thue system.
  2. An infinite sequence of configurations separated by special symbols $\#$.
  3. Let p be a prefix of a solution, then $|g(p)| < |h(p)|$.
  4. "Image under h is ahead of image under g by one configuration".

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- An infinite word $w \in dA^\omega$ is accepted by $\mathcal{A}^\gamma$ if and only if for some finite prefix p of w, $g(p) \not< h(p)$. Such a prefix p does exist for all infinite words except for the solutions of the instance (h, g). We call the verification of such a prefix p *error checking*.
- All infinite words beginning with letter in $A \setminus \{d\}$ are accepted by $\mathcal{A}^\gamma$.

First for each $a \in A$, let

$$\langle q_0, a, q_0, m(|h(a)| - |g(a)|)\rangle\,, \quad \langle q_1, a, q_1, m(-|g(a)|)\rangle\,, \quad \langle q_2, a, q_2, 0\rangle$$

be in σ, and for all $b \in A \setminus \{d\}$, let

$$\langle q_0, b, q_2, 0\rangle \in \sigma.$$

For error checking we need the following transitions for all letters $a \in A$: Let $h(a) = b_{j_1} b_{j_2} \cdots b_{j_{n_1}}$, where $b_{j_i} \in B$, for each index $1 \leq k \leq n_1$. Then let, for each $k = 1, \ldots, n_1$,

$$\langle q_0, a, q_1, m(k - |g(a)|) + j_k\rangle \in \sigma \tag{1}$$

Let $g(a) = b_{i_1} b_{i_2} \cdots b_{i_{n_2}}$, where $b_{i_\ell} \in B$, for each index $1 \leq \ell \leq n_2$. For each $\ell = 1, \ldots, n_2$ and letter $b_c \in B$ such that $b_{i_\ell} \neq b_c \in B$, let

$$\langle q_1, a, q_2, -m\ell - c\rangle \in \sigma. \tag{2}$$

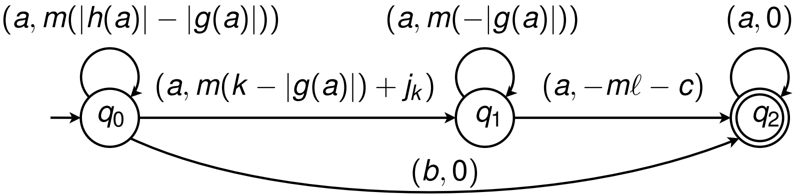
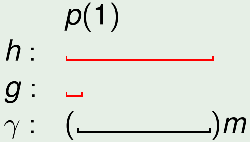
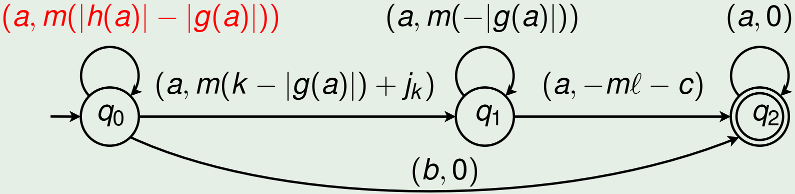


Figure: The weighted automaton $\mathcal{A}^\gamma$. In the figure $a \in A, b \in A \setminus \{d\}$ and $|A| = m - 1$.

Example

Let dw be an infinite word that is not a solution for ωPCP and let p be its prefix such that for some index r $h(p)(r) \neq g(p)(r)$.





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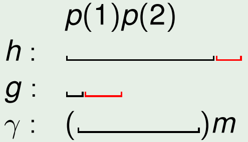

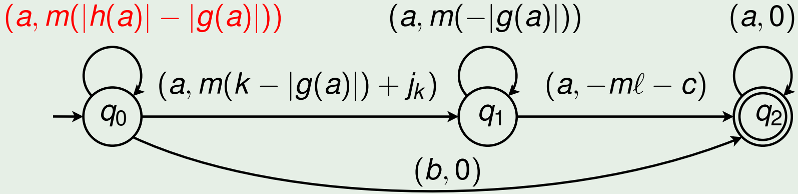

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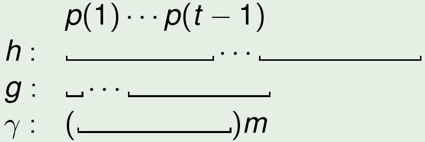

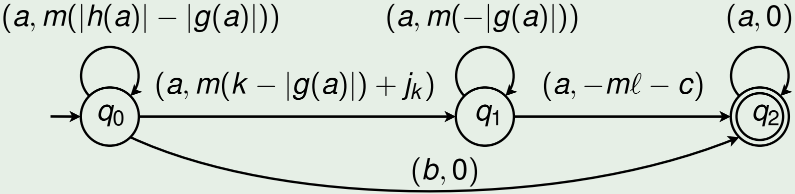

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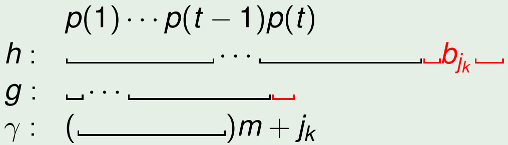

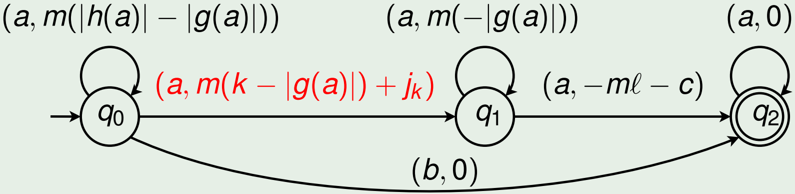

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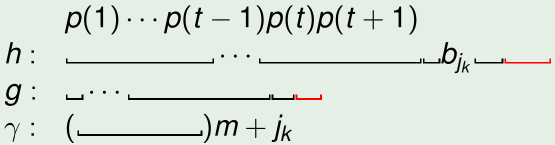

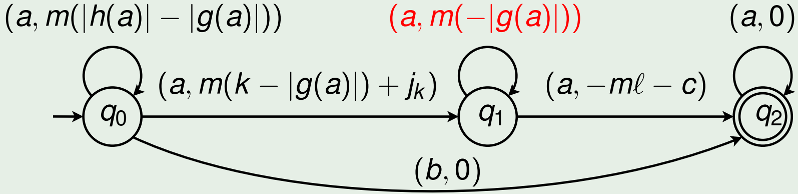

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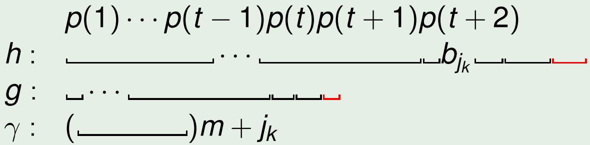

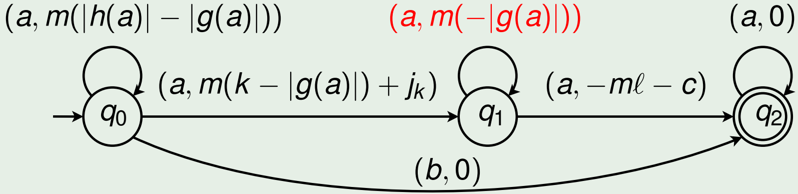

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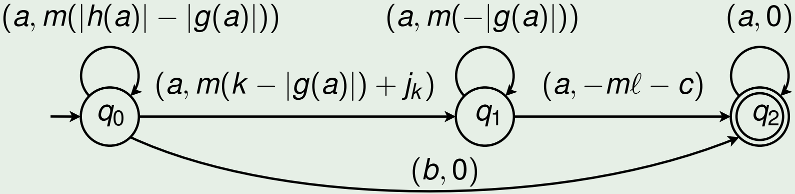
$p(1)\cdots p(t-1)p(t)p(t+1)\cdots p(n-1)$

h : ␣ $\cdots$ ␣ b_{j_k} ␣ ␣ $\cdots$ ␣

g : ␣ $\cdots$ ␣ ␣ ␣ $\cdots$ ␣

γ : $(␣)m + j_k$



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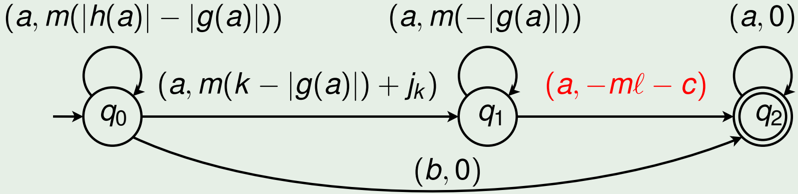

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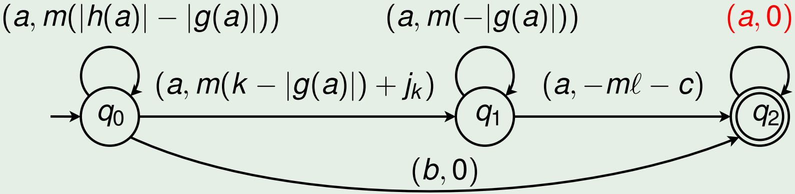

Theorem

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Corollary

It is undecidable whether or not L$(\mathcal{B}^\gamma) = A^\omega$ holds for 3-state integer weighted Büchi automata $\mathcal{B}^\gamma$, where each state is final, over its alphabet A.

- Robot Game with states is a two-player vector addition game on integer lattice $\mathbb{Z}^n$.
- From initial point $\mathbf{x}_0$, *Attacker* tries to reach the origin, while *Defender* tries to keep the game from reaching the origin.

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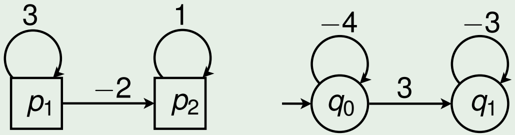
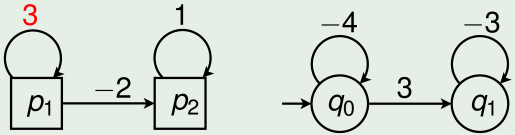
- From initial point $\mathbf{x}_0$, *Attacker* tries to reach the origin, while *Defender* tries to keep the game from reaching the origin.
- Vectors players can play are dependant on the state of the underlying automaton.
- The decision problem is whether there exists a winning strategy for Attacker. That is, can Attacker win, regardless of the vectors Defender plays.

Example



Consider one-dimensional game with initial point $x_0 = 2$.

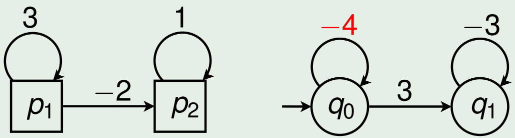
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$2 \to 5$

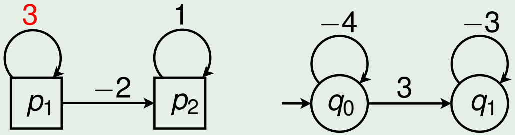
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$2 \rightarrow 5 \rightarrow 1$

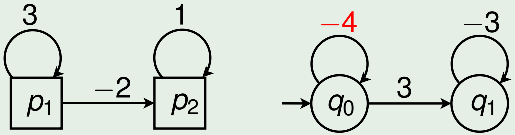
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$2 \to 5 \to 1 \to 4$

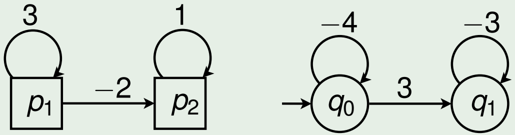
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$$2 \to 5 \to 1 \to 4 \to 0$$

Example



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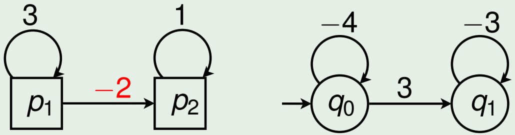
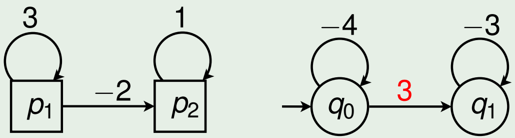


Consider one-dimensional game with initial point $x_0 = 2$.

$2 \rightarrow 5 \rightarrow 1 \rightarrow -1$

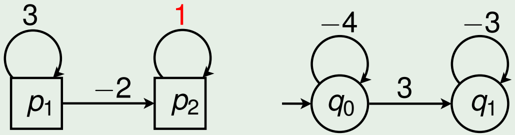
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$2 \to 5 \to 1 \to -1 \to 2$

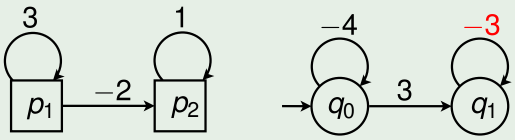
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$2 \to 5 \to 1 \to -1 \to 2 \to 3$

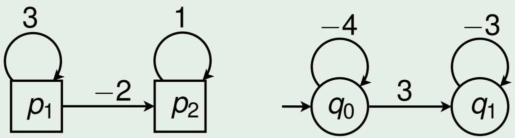
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$$2 \to 5 \to 1 \to -1 \to 2 \to 3 \to 0$$

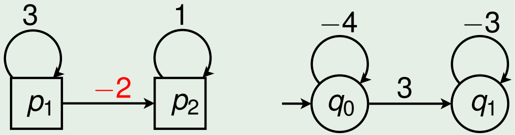
Example



Consider one-dimensional game with initial point $x_0 = 2$.

2

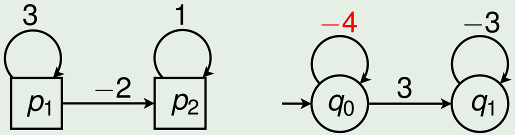
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$2 \to 0$

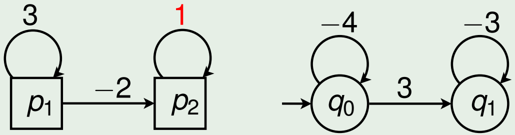
Example



Consider one-dimensional game with initial point $x_0 = 2$.

$2 \to 0 \to -4$

Example



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$2 \to 0 \to -4 \to -3$

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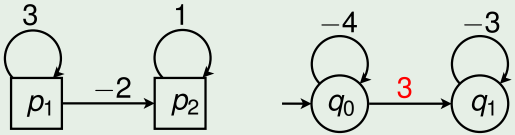

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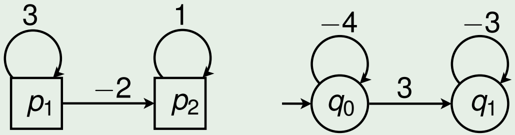

Consider one-dimensional game with initial point $x_0 = 2$.

$$2 \to 0 \to -4 \to -3 \to 0$$

$\therefore$ Attacker has a winning strategy.

- We construct a Robot Game with states where Attacker has a winning strategy if and only if the underlying automaton is universal.

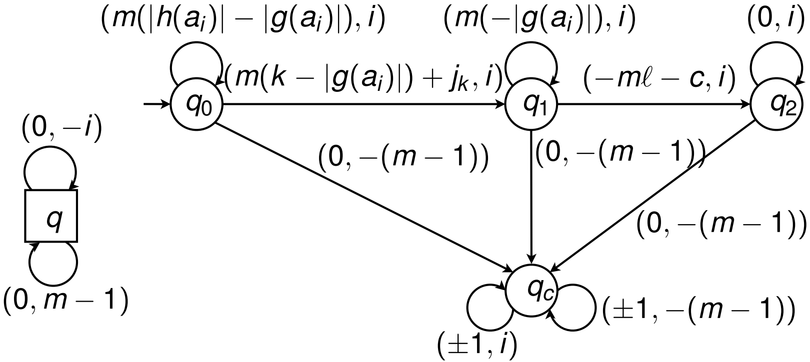

Figure: The picture of the weighted automata $\mathcal{B}$ and $\mathcal{A}$ with vectors substituting the transitions. In the figure $a_i \in A$.

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- If the word played by Defender is a solution of ωPCP, first component will never be 0.

Theorem

It is undecidable whether Attacker has a winning strategy in 2-dimensional Robot Game with states.

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Note that, while number of states is small for both players, for Defender number of edges is also small but for Attacker there are millions of edges.

- Universality Problem for 2-state integer weighted Büchi automata.
- An upper bound on number of edges in Robot Games with states.
- Application to similar Defender-Attacker games.

THANK YOU!