Combinatorial geometry and coding theory

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Combinatorial geometry: the space chromatic number

Definition (Nelson, 1950; Hadwiger, 1944).

The chromatic number of $\mathbb{R}^n$ is the minimum number $\chi(\mathbb{R}^n)$ of colors needed to color all the points in $\mathbb{R}^n$, so that any two points at the distance 1 apart receive different colors.

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- (Rogers, 1971) True for sets which are invariant under the actions of the symmetry group of a regular simplex.

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- Which exponent — n or $\sqrt{n}$, or another one — is true??

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Remark.

For (0,1)-vectors, "large enough" Hamming distance is the same as "small enough" scalar products. Thus, the problem is in finding the maximum cardinality of a set of (0,1)-vectors whose pairwise scalar products are small enough.

Three most important extremal values in coding theory

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Codes with forbidden Hamming distances.

Let $m(n, r, s)$ be the maximum cardinality of a binary code, in which any word has exactly r ones and any two words have scalar product not equal to s.

Where are the connections between our subjects?

Distance graphs.

Any graph $G = (V, E)$ with $V \subseteq \mathbb{R}^n$ and

$$E \subseteq \{\{\mathbf{x}, \mathbf{y}\} : \ |\mathbf{x} - \mathbf{y}| = a\}, \quad a > 0,$$

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Consider a sequence $G(n,r,s) = (V(n,r), E(n,r,s))$ of distance graphs with

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Eventually, we have:

$$\chi(\mathbb{R}^n) \geqslant \chi(G(n,r,s)) \geqslant \frac{|V(n,r)|}{\alpha(G(n,r,s))} = \frac{\binom{n}{r}}{m(n,r,s)}.$$

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Bit more difficult connections for Borsuk's problem. Anyway, one has to find $m(n, r, s)$ and $\chi(n, r, s) = \chi(G(n, r, s))$.

The value $m(n, r, s)$

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Theorem (Frankl, Füredi, 1985).

If $r \geqslant 2s + 1$, then

$$m(n, r, s) = \binom{n-s-1}{r-s-1} \sim \frac{n^{r-s-1}}{(r-s-1)!}.$$

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Theorem (Frankl, Wilson, 1981, and Rödl, 1985).

If $r < 2s + 1$ and $r - s$ is a prime power, then

$$m(n,r,s) \sim n^s \frac{(2r-2s-1)!}{r!(r-s-1)!}.$$

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Questions.

- What's with the case when $r - s$ is not a prime power in the second theorem?

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$$m(n, r, s) = \binom{n - s - 1}{r - s - 1} \sim \frac{n^{r-s-1}}{(r - s - 1)!}.$$

Theorem (Frankl, Wilson, 1981, and Rödl, 1985).

If $r < 2s + 1$ and $r - s$ is a prime power, then

$$m(n, r, s) \sim n^s \frac{(2r - 2s - 1)!}{r!(r - s - 1)!}.$$

Questions.

- What's with the case when $r - s$ is not a prime power in the second theorem?
- How to find the exact value in the second theorem?

The value $m(n, r, s)$

Theorem (Nagy, 1972).

If $n \equiv 0 \pmod 4$, then $m(n, 3, 1) = n$. If $n \equiv 1 \pmod 4$, then $m(n, 3, 1) = n - 1$. If $n \equiv 2, 3 \pmod 4$, then $m(n, 3, 1) = n - 2$.

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One has $m(n, r, r-1) \sim \frac{n^{r-1}}{r!}$.

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Note that the graph $G(n, r, 0)$ is the classical Kneser graph.

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Theorem (using Frankl, Füredi, 1985, and Turán numbers).

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Theorem (Balogh, Kostochka, Raigorodskii).

If $n = 2^k$, then $\chi(n, 3, 1) = \frac{(n-1)(n-2)}{6}$. Anyway, $\chi(n, 3, 1) \sim \frac{n^2}{6}$.

The value $\chi(n, r, s)$

Theorem (Bobu, Kostina, Kupriyanov, 2014+).

If $n = 2^k$, then

$$n - r + 1 \leqslant \chi(n, r, r-1) \leqslant n.$$

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Theorem (Bobu, Kostina, Kupriyanov, 2014+).

One has

$$\frac{n^2}{6}(1 + o(1)) \leqslant \chi(n, 4, 2) \leqslant \frac{n^2}{2}(1 + o(1)).$$

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Theorem (Lovász, 1975).

One has $\chi(n, r, 0) = n - 2r + 2$.

This result is the classical proof of Kneser's conjecture.

Random subgraphs of $G(n, r, s)$

Let $p = p(n) \in [0, 1]$. Let $G_p(n, r, s)$ be a random element taking values in the set of all spanning subgraphs $G = (V(n, r), E)$ of the graph $G(n, r, s)$ with binomial distribution

$$\mathbb{P}(G_p(n, r, s) = G) = p^{|E|}(1 - p)^{|E(n,r,s)|-|E|},$$

i.e., any edge of $G(n, r, s)$ belongs to $G_p(n, r, s)$ with probability p independently of all the other edges.

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The main question.

Let $m_p(n, r, s) = \alpha(G_p(n, r, s))$, $\chi_p(n, r, s) = \chi(G_p(n, r, s))$. How, with high probability, do these quantities differ from the original ones?

Stability of $m_p(n, r, 0)$

Theorem (Bollobás, Narayanan, AMR, 2014+).

Fix an $\varepsilon > 0$ and let $r = r(n)$ be a natural number such that $2 \leqslant r = o\left(n^{1/3}\right)$. Then

$$\mathbb{P}\left(m_p(n, r, 0) = m(n, r, 0)\right) \to 1,$$

provided $p \geqslant (1+\varepsilon)p_c(n, r)$, where

$$p_c(n, r) = \frac{(r+1)\ln n - r\ln r}{\binom{n-1}{r-1}}.$$

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An absolutely incredible stability! What's with other values r, s?

Other quantities $m_p(n, r, s)$

Theorem (Bogoliubski, Gusev, Pyaderkin, AMR, 2014+).

If $r \leqslant 2s + 1$, then, w.h.p.,

$$m_{1/2}(n, r, s) \asymp m(n, r, s) \ln n.$$

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For example, if $r \geqslant 2$ and $s = 0$, then this theorem is a weakened version of the Bollobás–Narayanan–AMR theorem. Together with the first theorem of this slide, it sais that we have a kind of "phase transition" when coming from $r \leqslant 2s + 1$ to $r > 2s + 1$.

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Both theorems are true for a much larger range of values p.

Quantities $\chi_p(n, r, s)$

Theorem (Kupavskiy, 2014+).

If p is constant and $2 \leqslant r \ll \frac{n}{2}$, then. w.h.p., $\chi_p(n, r, 0) \sim \chi(n, r, 0)$.

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Question.

What is the asymptotics of the value $\chi(n, r, 0) - \chi_p(n, r, 0)$?

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The same is true for many other values of p.