Asymmetry in Discrete–Time Bioresource Management Problem

A.N. Rettieva

Institute of Applied Mathematical Research
Karelian Research Center of RAS
Petrozavodsk

OUTLINE

1. History of the models of "fish wars"

2. Model with asymmetric players and the Nash equilibrium

3. Cooperative behavior and the Nash bargaining procedure

4. Model with fixed harvesting times

5. Model with random harvesting times

1. History of the models of "fish wars"

Levhari and Mirman (1980)

The biological growth rule is given by

$$x_{t+1} = (x_t)^{\alpha}, \;\; x_0 = x,$$

where $x_t \geq 0$ – size of the population, $0 < \alpha < 1$ – natural birth rate.

Two players exploit this fish stock and the utility functions are logarithmic.

Then the players' net revenue over infinite time horizon are:

$$\bar{J}_i = \sum_{t=0}^{\infty} \beta_i^t \ln(u_t^i),$$

where $u_t^i \geq 0$ – players' catch at time t, $0 < \beta_i < 1$ – the discount factor for player i.

And the dynamic becomes

$$x_{t+1} = (x_t - u_t^1 - u_t^2)^\alpha\,, \;\; x_0 = x\,.$$

Authors derived Cournot-Nash and cooperative equilibria.

Our model with many players

The dynamics of the fishery is described by the equation

$$x_{t+1} = (\varepsilon x_t - \sum_{i=1}^{n} u_{it})^{\alpha}\,, \;\; x_0 = x\,,$$

where $x_t \geq 0$ – size of population at a time t, $\varepsilon \in (0,1)$ – natural death rate, $\alpha \in (0,1)$ – natural birth rate, $u_{it} \geq 0$ – the catch of player i, $i = 1, \ldots, n$.

The characteristic function for cooperative game is constructed in two unusual forms, and we derived Shapley value and time-consistent imputation distribution procedure.

Fisher and Mirman (1992)

The biological growth rule is given by

$$x_{t+1} = f((x_t - c_{1t}), (y_t - c_{2t}))\,,$$
$$y_{t+1} = g((x_t - c_{1t}), (y_t - c_{2t}))\,,$$

where $x_t \geq 0$ – size of the population in the first region, $y_t \geq 0$ – size of the population in the second region, $0 \leq c_{1t} \leq x_t$, $0 \leq c_{2t} \leq y_t$ – players' catch at time t.

Players wish to maximize the sum of discounted utility

$$\sum_{t=1}^{\infty} \delta_1^t \ln(c_{1t}), \quad \sum_{t=1}^{\infty} \delta_2^t \ln(c_{2t})\,,$$

where $0 < \delta_i < 1$ – the discount factors ($i = 1, 2$).

Our model of bioresource sharing problem

The center (referee) shares a reservoir between the competitors. There are migratory exchanges between the regions of the reservoir.

The dynamics is of the form

$$\begin{cases} x_{t+1} = (x_t - u_{1t})^{\alpha_1 - \beta_1 s} (y_t - u_{2t})^{\beta_1 s}\,, \\ y_{t+1} = (y_t - u_{2t})^{\alpha_2 - \beta_2 (1-s)} (x_t - u_{1t})^{\beta_2 (1-s)}\,, \end{cases}$$

where $x_t \geq 0$ – size of the population in the first region, $y_t \geq 0$ – size of the population in the second region, $0 < \alpha_i < 1$ – natural birth rate, $0 < \beta_i < 1$ – coefficients of migration between the regions ($i = 1,2$), $0 \leq u_{1t} \leq x_t$, $0 \leq u_{2t} \leq y_t$ – countries' catch at time t, $0 < \delta_i < 1$ – the discount factor for country i ($i = 1,2$).

Here the intensity of migration also depends on the share s. It seems to be natural because the habitat size decreases as s decreases and fish needs to migrate to another region.

We consider the problem of maximizing infinite sum of discounted utilities for two players:

$$J_1 = \sum_{t=0}^{\infty} \delta_1^t \ln(u_{1t}) \, , \quad J_2 = \sum_{t=0}^{\infty} \delta_2^t \ln(u_{2t}) \, .$$

We derived Nash, cooperative and incentive equilibria.

2. The model with asymmetric players and the Nash equilibrium

Two players exploit the fish stock. The dynamics of the fishery is

$$x_{t+1} = (\varepsilon x_t - u_{1t} - u_{2t})^{\alpha}\,, \;\; x_0 = x\,, \tag{1}$$

where $x_t \geq 0$ – the size of population at a time t, $\varepsilon \in (0, 1)$ – natural survival rate, $\alpha \in (0, 1)$ – natural birth rate, $u_{it} \geq 0$ – the catch of player i, $i = 1, 2$.

The players' net revenues over time interval [0,n] are

$$J_i = \sum_{t=0}^{n} \delta_i^t \ln(u_{it})\,, \tag{2}$$

where $0 < \delta_i < 1$ – the discount factor for country i, $i = 1, 2$.

(u_1^N, u_2^N) – Nash equilibrium if

$$J_1(u_1^N, u_2^N) \geq J_1(u_1, u_2^N)\,,\ J_2(u_1^N, u_2^N) \geq J_2(u_1^N, u_2)\,,\ \forall u_1, u_2\,.$$

The Nash equilibrium of the problem (1), (2) is

$$u_1^{Nn} = \frac{\varepsilon a_2 \sum\limits_{j=0}^{n-1} a_1^j}{\sum\limits_{j=0}^{n} a_1^j \sum\limits_{j=0}^{n} a_2^j - 1}\,, \quad u_2^{Nn} = \frac{\varepsilon a_1 \sum\limits_{j=0}^{n-1} a_2^j}{\sum\limits_{j=0}^{n} a_1^j \sum\limits_{j=0}^{n} a_2^j - 1}\,,$$

where $a_i = \alpha\delta_i\,,\ i = 1, 2$.

The payoffs for n-stage game are

$$V_i^N(x, \delta_i) = \sum_{j=0}^{n} (a_i)^j \ln(x) + \sum_{j=1}^{n} (\delta_i)^{n-j} A_i^j - (\delta_i)^n \ln(2)\,, \qquad (3)$$

where

$$A_1^j = \ln\Big[\Big(\frac{\varepsilon \sum_{k=1}^{j} a_2^k}{\sum_{k=0}^{j-1} a_1^k \sum_{k=0}^{j-1} a_2^k - 1}\Big)^{\sum_{k=0}^{j} a_1^k}\Big(\sum_{k=1}^{j} a_1^k\Big)^{\sum_{k=1}^{j} a_1^k}\Big],$$

$$A_2^j = \ln\Big[\Big(\frac{\varepsilon \sum_{k=1}^{j} a_1^k}{\sum_{k=0}^{j-1} a_1^k \sum_{k=0}^{j-1} a_2^k - 1}\Big)^{\sum_{k=0}^{j} a_2^k}\Big(\sum_{k=1}^{j} a_2^k\Big)^{\sum_{k=1}^{j} a_2^k}\Big].$$

The size of the stock after n periods has the form

$$x^{Nn} = x_0^{\alpha^n}(\varepsilon a_1 a_2)^{\sum_{j=1}^{n} \alpha^j} \prod_{l=1}^{n}\left(\frac{\sum_{j=0}^{l-1} a_1^j \sum_{j=0}^{l-1} a_2^j}{\sum_{j=0}^{l} a_1^j \sum_{j=0}^{l} a_2^j - 1}\right)^{\alpha^{n-l+1}}. \qquad (4)$$

3. Cooperative behavior and the Nash bargaining procedure

Here we obtain the cooperative strategies without determining the joint discount factor using recursive Nash bargaining procedure. On each time moment the cooperative strategies are determined as the Nash bargaining solution taking the non-cooperative profits as a status-quo point.

We start with the one-step game and assume that if there were no future period, the countries would get the remaining fish in the ratio $1:1$. Let the initial size of the population be x.

Noncooperative gains are

$$H_1^{1N} = (1+a_1)\ln(x) + A_1^1 - \delta_1 \ln(2)\,, \quad (5)$$

$$H_2^{1N} = (1+a_2)\ln(x) + A_2^1 - \delta_2 \ln(2)\,, \quad (6)$$

where A_1^1 and A_2^1 are independent on x and have the forms

$$A_1^1 = \ln \frac{(\varepsilon a_2)^{1+a_1} a_1^{a_1}}{((1+a_1)(1+a_2)-1)^{1+a_1}},\; A_2^1 = \ln \frac{(\varepsilon a_1)^{1+a_2} a_2^{a_2}}{((1+a_1)(1+a_2)-1)^{1+a_2}}.$$

The cooperative strategies are determined maximizing the Nash product

$$\begin{aligned} H^{1c} = (\ln(u_1) + a_1 \ln(\varepsilon x - u_1 - u_2) - \delta_1 \ln(2) - H_1^{1N}) \cdot \\ \cdot(\ln(u_2) + a_2 \ln(\varepsilon x - u_1 - u_2) - \delta_2 \ln(2) - H_2^{1N}) = \\ = (H_1^{1c} - H_1^{1N})(H_2^{1c} - H_2^{1N}) \to \max\,, \end{aligned}$$

where H_i^{1N} are given in (5)–(6).

The cooperative strategies are

$$u_1 = \gamma_1^{1c} x\,,\; u_2 = \gamma_2^{1c} x\,,$$

and can be found as the solution of the next equation

$$\gamma_2^{1c}\left(\ln(\gamma_2^{1c})+a_2\ln(\varepsilon-\gamma_1^{1c}-\gamma_2^{1c})-A_2^1\right)=\gamma_1^{1c}\left(\ln(\gamma_1^{1c})+a_1\ln(\varepsilon-\gamma_1^{1c}-\gamma_2^{1c})-A_1^1\right) \tag{7}$$

with the relation

$$\gamma_2^{1c}=\frac{\varepsilon-\gamma_1^{1c}(1+a_1)}{1+a_2}.$$

The cooperative gains for one step game have the forms

$$H_1^{1c}=(1+a_1)\ln(x)+\ln(\gamma_1^{1c})+a_1\ln(\varepsilon-\gamma_1^{1c}-\gamma_2^{1c})-\delta_1\ln(2), \quad (8)$$

$$H_2^{1c}=(1+a_2)\ln(x)+\ln(\gamma_2^{1c})+a_2\ln(\varepsilon-\gamma_1^{1c}-\gamma_2^{1c})-\delta_2\ln(2). \quad (9)$$

We pass to two stage game. If the players act non-cooperatively till the end of the game then the gains are

$$H_1^{2N} = (1 + a_1 + a_1^2)\ln(x) + A_1^2 + \delta_1 A_1^1 - \delta_1^2 \ln(2)\,, \tag{10}$$

$$H_2^{2N} = (1 + a_2 + a_2^2)\ln(x) + A_2^2 + \delta_2 A_2^1 - \delta_2^2 \ln(2)\,, \tag{11}$$

where

$$A_1^2 = \ln \frac{(\varepsilon(a_2 + a_2^2))^{1+a_1+a_1^2}(a_1 + a_1^2)^{a_1+a_1^2}}{((1 + a_1)(1 + a_2) - 1)^{1+a_1+a_1^2}}\,,$$

$$A_2^2 = \ln \frac{(\varepsilon(a_1 + a_1^2))^{1+a_2+a_2^2}(a_2 + a_2^2)^{a_2+a_2^2}}{((1 + a_1)(1 + a_2) - 1)^{1+a_2+a_2^2}}\,.$$

We determine the cooperative strategies maximizing the Nash product

$$H^{2c} = (\ln(u_1) + \delta_1 H_1^{1c} - H_1^{2N})(\ln(u_2) + \delta_2 H_2^{1c} - H_2^{2N}) = \\ = (H_1^{2c} - H_1^{2N})(H_2^{2c} - H_2^{2N}\Big) \to \max\,,$$

where H_i^{1c} are the cooperative gains for one step game and are given in (8)–(9) and H_i^{2N} are determined in (10)–(11).

Analogously we get the equation for γ_1^{2c} and γ_2^{2c}

$$\gamma_2^{2c}\Big(\ln(\gamma_2^{2c}) + (a_2 + a_2^2)\ln(\varepsilon - \gamma_1^{2c} - \gamma_2^{2c}) +$$
$$+\delta_2(\ln(\gamma_2^{1c}) + a_2 \ln(\varepsilon - \gamma_1^{1c} - \gamma_2^{1c})) - A_2^2 - \delta_2 A_2^1\Big) =$$
$$\gamma_1^{2c}\Big(\ln(\gamma_1^{2c}) + (a_1 + a_1^2)\ln(\varepsilon - \gamma_1^{2c} - \gamma_2^{2c}) +$$
$$+\delta_1(\ln(\gamma_1^{1c}) + a_1 \ln(\varepsilon - \gamma_1^{1c} - \gamma_2^{1c})) - A_1^2 - \delta_1 A_1^1\Big) \qquad (12)$$

with the relation

$$\gamma_2^{2c} = \frac{\varepsilon - \gamma_1^{2c}(1 + a_1 + a_1^2)}{1 + a_2 + a_2^2}.$$

The process can be repeated for the n-stage game and we have the next form of the cooperative profits

$$H_1^{nc}(\gamma_1^1,\ldots,\gamma_1^n,\gamma_2^1,\ldots,\gamma_2^n)=\sum_{j=0}^{n}a_1^j\ln(x)+$$

$$\sum_{j=0}^{n-1}\delta_1^{n-j}\Big[\ln(\gamma_1^{(n-j)c})+\sum_{i=1}^{n-j}a_1^i\ln(\varepsilon-\gamma_1^{(n-j)c}-\gamma_2^{(n-j)c})\Big]-\delta_1^n\ln(2)(13)$$

and

$$H_2^{nc}(\gamma_1^1,\ldots,\gamma_1^n,\gamma_2^1,\ldots,\gamma_2^n)=\sum_{j=0}^{n}a_2^j\ln(x)+$$

$$\sum_{j=0}^{n-1}\delta_2^{n-j}\Big[\ln(\gamma_2^{(n-j)c})+\sum_{i=1}^{n-j}a_2^i\ln(\varepsilon-\gamma_1^{(n-j)c}-\gamma_2^{(n-j)c})\Big]-\delta_2^n\ln(2).(14)$$

The cooperative strategies can be found recursively from the equations

$$\gamma_2^{nc}\Big(\sum_{j=0}^{n-1}\delta_2^{n-j}\Big[\ln(\gamma_2^{(n-j)c})+\sum_{i=1}^{n-j}a_2^i\ln(\varepsilon-\gamma_1^{(n-j)c}-\gamma_2^{(n-j)c})\Big]-\delta_2^jA_2^{n-j}\Big)=$$

$$\gamma_1^{nc}\Big(\sum_{j=0}^{n-1}\delta_1^{n-j}\Big[\ln(\gamma_1^{(n-j)c})+\sum_{i=1}^{n-j}a_1^i\ln(\varepsilon-\gamma_1^{(n-j)c}-\gamma_2^{(n-j)c})\Big]-\delta_1^jA_1^{n-j}\Big)$$

with the relation

$$\gamma_2^{nc}=\frac{\varepsilon-\gamma_1^{nc}\sum\limits_{i=0}^{n}a_1^i}{\sum\limits_{i=0}^{n}a_2^i}.$$

Modelling

We present the results of numerical modelling for 20-stage game with the next parameters:

$$\varepsilon = 0.6\,, \quad \alpha = 0.3\,, \quad x_0 = 0.8\,,$$
$$\delta_1 = 0.85\,, \quad \delta_2 = 0.9\,.$$

The cooperative and Nash gains are

$$V_1^{nc}(x, \delta_1) = -14.1039 > V_1^N(x, \delta_1) = -14.6439\,,$$

$$V_2^{nc}(x, \delta_2) = -20.5108 > V_2^N(x, \delta_2) = -23.2596\,.$$

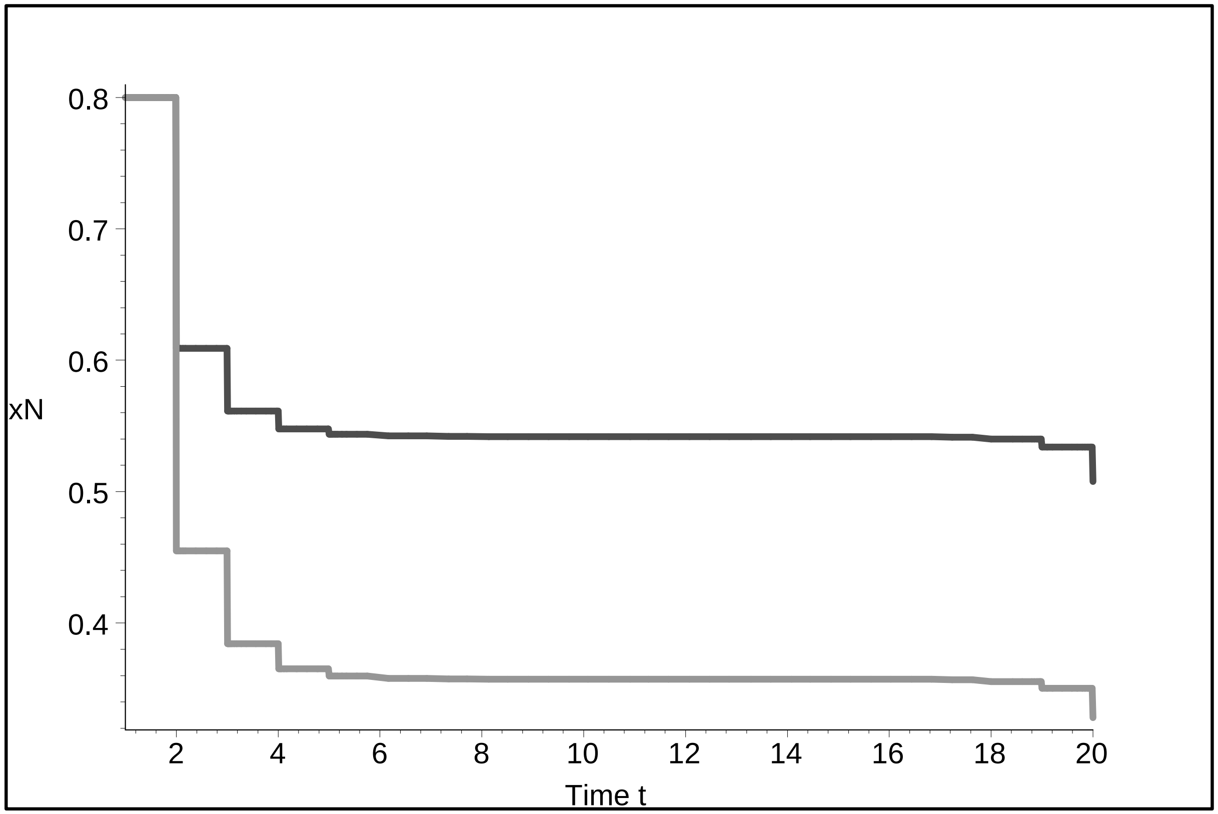

Fig. 1. The population size: dark – cooperative, light – Nash

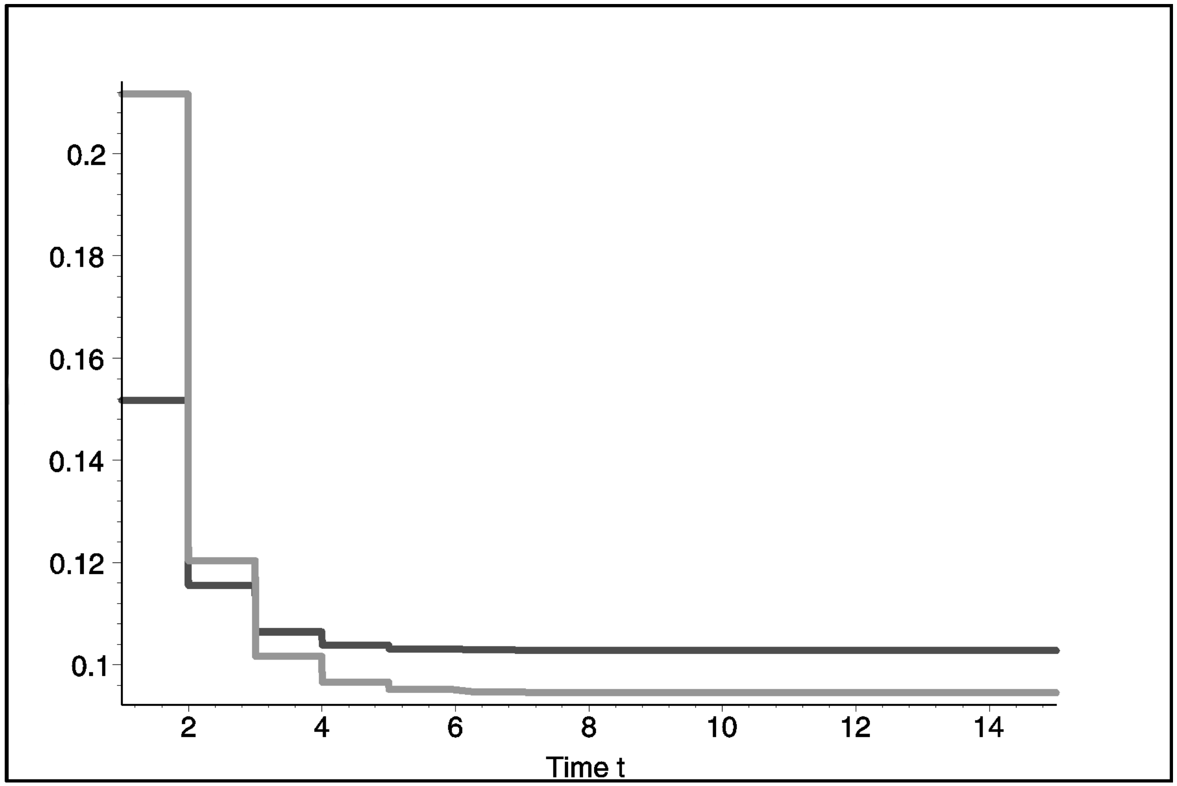

Fig. 2. The catch of player 1: dark – cooperative, light – Nash

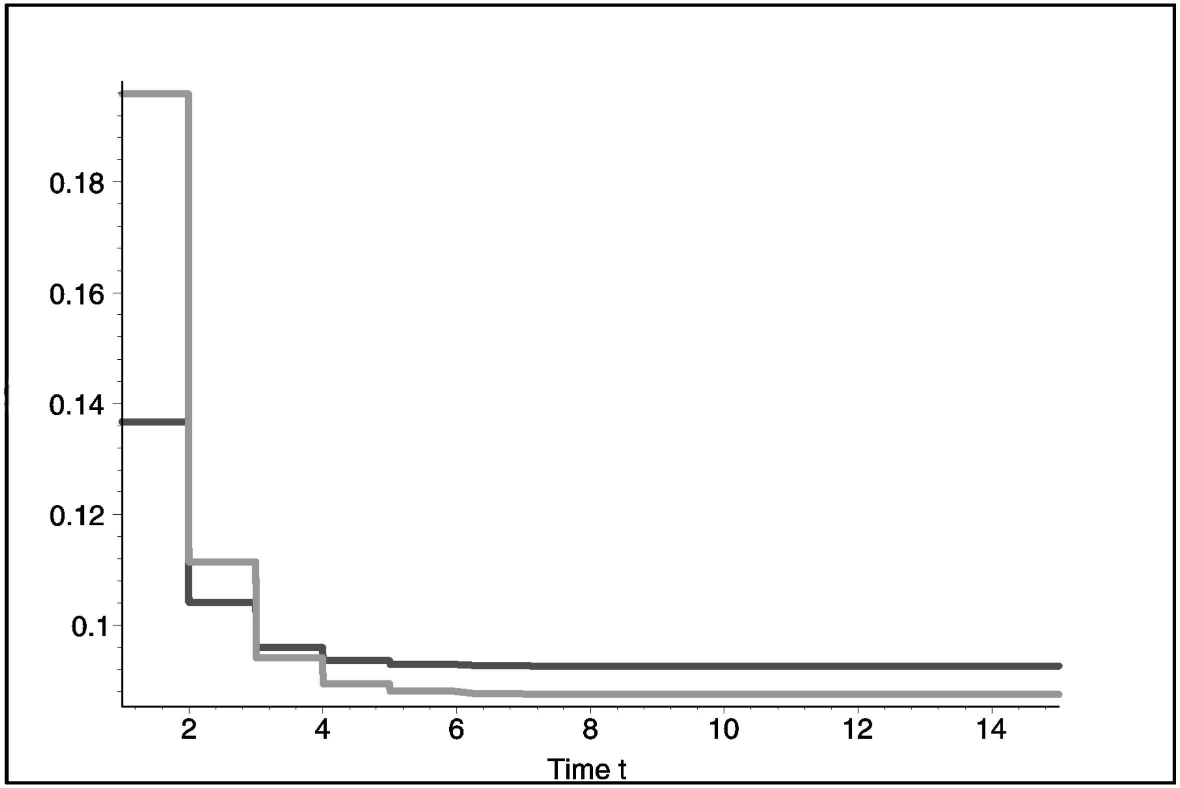

Fig. 3. The catch of player 2: dark – cooperative, light – Nash

Next we compare players' profits for different discount factors. In Fig. 4 we show $V_1^{nc}(x, \delta_1)$ and $V_2^{nc}(x, \delta_2)$ for $\delta_1 = 0.1 \ldots 0.9$ and $\delta_2 = 0.1 \ldots 0.9$. One can notice that player with greater discount factor gets more advantage from cooperative agreement. And both players get equal profits when their discount factors coincide.

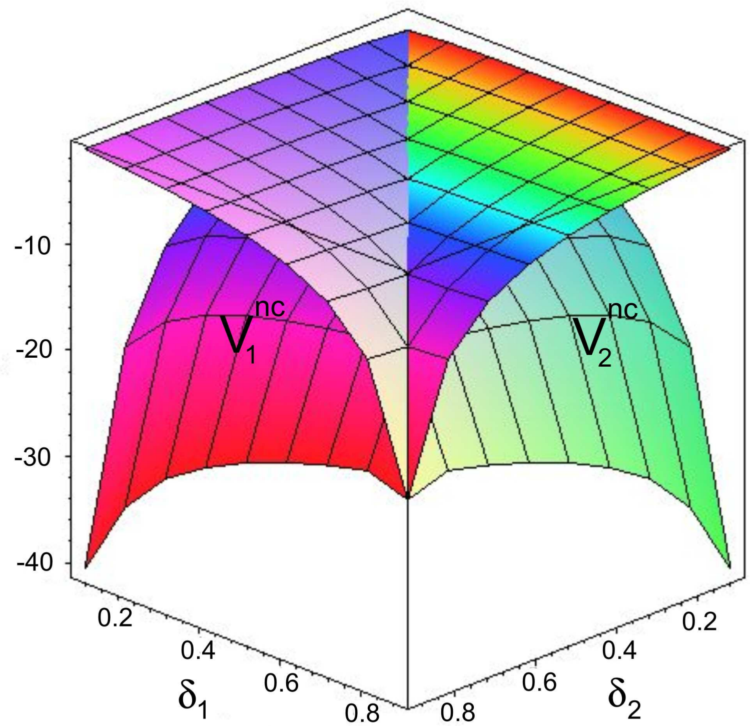

Fig. 4. The catch of player 2: dark – cooperative, light – Nash

One can notice that our approach gives a player the payoff that is greater or equal (for some parameters) Nash payoff. In Fig. 5 it is shown the second player's payoff for different discount factors. So it proves that under presented approach it is always profitable to cooperate versus the maximization the weighted sum where players can get less under cooperation.

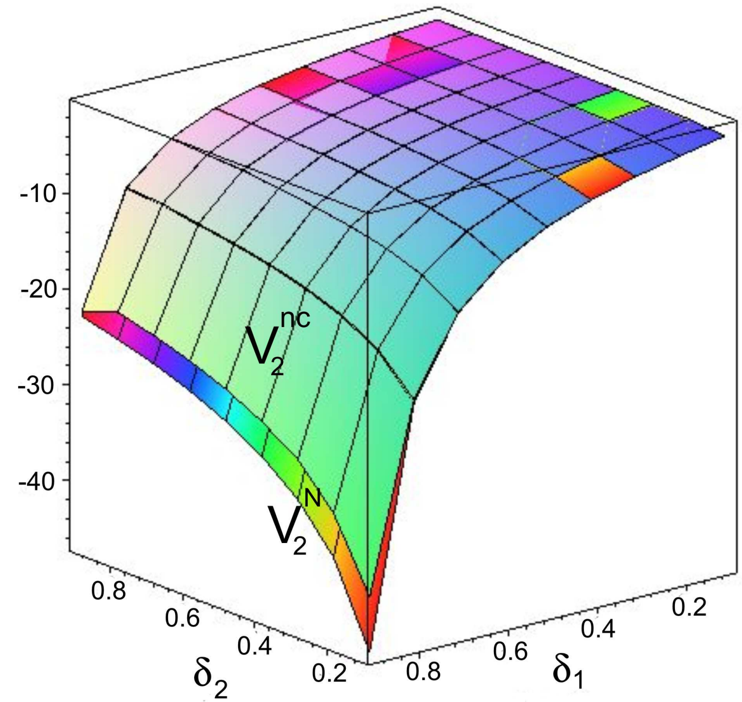

Fig. 5. The catch of player 2: dark – cooperative, light – Nash

4. Model with fixed harvesting times

Let us consider the case where the first player harvests the stock for n_1 time moments, and the second – for n_2. Let $n_1 < n_2$. So, we have a situation where on time interval $[0, n_1]$ players cooperate, and we need to determine their strategies. After n_1 and until n_2 the second player acts individually. So, the players' profits have the forms

$$J_1 = \sum_{t=0}^{n_1} \delta_1^t \ln(u_{1t}^c)\,,\ J_2 = \sum_{t=0}^{n_1} \delta_2^t \ln(u_{2t}^c) + \sum_{t=n_1+1}^{n_2} \delta_2^t \ln(u_{2t}^a)\,, \quad (15)$$

where u_i^c, $i = 1,2$ – cooperative strategies, u_2^a – the second player's strategy when harvesting the stock alone.

We construct cooperative strategies and the joint payoff maximizing the Nash product for the whole game:

$$
\begin{aligned}
&(V_1^c(x,\delta_1)[0,n_1] - V_1^N(x,\delta_1)[0,n_1]) \cdot \\
&\cdot(V_2^c(x,\delta_2)[0,n_1] + V_2^{ac}(x^{cn_1},\delta_2)[n_1,n_2] - \\
&-V_2^N(x,\delta_2)[0,n_1] - V_2^{aN}(x^{Nn_1},\delta_2)[n_1,n_2]) = \\
&= (\sum_{t=0}^{n_1} \delta_1^t \ln(u_{1t}^c) - V_1^N(x,\delta_1)[0,n_1]) \cdot \\
&\cdot(\sum_{t=0}^{n_1} \delta_2^t \ln(u_{2t}^c) + \sum_{t=n_1+1}^{n_2} \delta_2^t \ln(u_{2t}^a) - \\
&-V_2^N(x,\delta_2)[0,n_1] - V_2^{aN}(x^{Nn_1},\delta_2)[n_1,n_2]) \to \max,
\end{aligned}
\tag{16}
$$

where $V_i^N(x,\delta_i)[0,n_1]$ are the non-cooperative gains determined in (3) (where $n = n_1$), $V_2^{ac}(x^{cn_1},\delta_2)[n_1,n_2]$ – the second player's gain when acting individually after n_1 periods of cooperation,

$V_2^{aN}(x^{Nn_1}, \delta_2)[n_1, n_2]$ – the second player's gain when acting individually after n_1 periods of noncooperation.

Let us consider the time interval $[n_1, n_2]$, here the second player acts individually.

After $n = n_2 - n_1$ steps we get $\gamma_{2n} = \frac{\varepsilon}{\sum\limits_{j=0}^{n} a_2^j}$ and the payoff

$$V_2^{ac}(x^{cn_1}, \delta_2)[n_1, n_2] = H_2^n(\gamma_{21}^c, \ldots, \gamma_2^n) = \sum_{j=0}^{n} a_2^j \ln x + \sum_{j=1}^{n} \delta_2^{n-j} B^j \,, \tag{17}$$

where

$$B^j = \sum_{l=0}^{j} a_2^l \ln(\frac{\varepsilon}{\sum\limits_{p=0}^{j} a_2^p}) + \sum_{l=1}^{j} a_2^l \ln(\sum_{p=1}^{j} a_2^p) \,, \; n = n_2 - n_1 \,.$$

Now we can determine $V_2^{aN}(x^{Nn_1}, \delta_2)[n_1, n_2]$ as the second player's payoff starting from the noncooperative point x^{Nn_1} (see (4) when $n = n_1$):

$$V_2(x, \delta_2)[n_1, n_2] = \sum_{j=0}^{n} a_2^j \ln(x^{Nn_1}) + \sum_{j=1}^{n} \delta_2^{n-j} B^j \,.$$

So, we determine all the gains in problem (16) except for the cooperative gains. To find the later we determine cooperative payoffs for an n_1-step game starting from n_1. Be reminded that after n_1 we assume that the first player gets a portion of the remaining stock – k and the second player starts exploiting the portion $(1-k)$ of the remaining stock.

We start with the one-step game. As usual we seek the players strategies in the linear form $u_{11}^c = \gamma_1^1 x$ and $u_{21}^c = \gamma_2^1 x$.

Then the first player's profit is

$$
\begin{aligned}
&H_{11}^c(\gamma_{11}^c, \gamma_{21}^c; x) = \\
&= \ln(\gamma_{11}^c x) + \delta_1 \ln(k(\varepsilon x - \gamma_{11}^c x - \gamma_{21}^c x)^{\alpha}) = \\
&= (1 + a_1)\ln(x) + \ln(\gamma_{11}^c) + a_1 \ln(\varepsilon - \gamma_{11}^c - \gamma_{21}^c) + \delta_1 \ln(k)
\end{aligned}
$$

and the second player's profit is

$$H_{21}^c(\gamma_{11}^c, \gamma_{21}^c; x) = \ln(\gamma_{21}^c x) + \delta_2 V_2^{ac}(x^{cn_1}, \delta_2)[n_1, n_2] =$$
$$= \ln(\gamma_{21}^c x) + \delta_2 \sum_{j=0}^{n} a_2^j \ln((1-k)(\varepsilon x - \gamma_{11}^c x - \gamma_{21}^c x)^\alpha) + \sum_{j=1}^{n} \delta_2^{n+1-j} B^j =$$
$$= \ln(\gamma_{21}^c) + \sum_{j=0}^{n+1} a_2^j \ln x + \sum_{j=1}^{n+1} a_2^j \ln(\varepsilon - \gamma_{11}^c - \gamma_{21}^c) +$$
$$+ \sum_{j=1}^{n} \delta_2^{n+1-j} B^j + \delta_2 \sum_{j=0}^{n} a_2^j \ln(1-k)\,.$$

We can now consider problem (16) for the two-step game. The objective function of the first player for the two-step game is

$$
\begin{aligned}
& H^c_{12}(\gamma^c_{11}, \gamma^c_{12}, \gamma^c_{12}, \gamma^c_{22}; x) = \\
& = \ln(\gamma^c_{12} x) + \delta_1 H^{1c}_1(\gamma^c_{11}, \gamma^c_{21}; (\varepsilon x - \gamma^c_{12} x - \gamma^c_{22} x)^\alpha) = \\
& = \ln(\gamma^c_{12} x) + \delta_1 (1 + a_1) \ln(\varepsilon x - \gamma^c_{12} x - \gamma^c_{22} x)^\alpha + \\
& + \delta_1 (\ln(\gamma^c_{11}) + a_1 \ln(\varepsilon - \gamma^c_{11} - \gamma^c_{21}) + \delta_1 \ln(k)) = \\
& = (1 + a_1 + a_1^2) \ln(x) + \ln(\gamma^c_{12}) + a_1 (1 + a_1) \ln(\varepsilon - \gamma^c_{12} - \gamma^c_{22}) + \\
& + \delta_1 \ln(\gamma^c_{11}) + \delta_1 a_1 \ln(\varepsilon - \gamma^c_{11} - \gamma^c_{21}) + \delta_1^2 \ln(k) \,,
\end{aligned}
$$

and that of the second player

$$
\begin{aligned}
&H^c_{22}(\gamma^c_{11}, \gamma^c_{21}, \gamma^c_{12}, \gamma^c_{22}; x) = \\
&= \ln(\gamma^c_{22} x) + \delta_2 H^{1c}_2(\gamma^c_{11}, \gamma^c_{21}; (\varepsilon x - \gamma^c_{12} x - \gamma^c_{22} x)^\alpha) = \\
&= \ln(\gamma^c_{22} x) + \delta_2 \ln(\gamma^c_{21}) + \delta_2 \sum_{j=0}^{n+1} a_2^j \ln(\varepsilon x - \gamma^c_{12} x - \gamma^c_{22} x)^\alpha + \\
&+ \delta_2 \sum_{j=1}^{n+1} a_2^j \ln(\varepsilon - \gamma^c_{11} - \gamma^c_{21}) + \sum_{j=1}^{n} \delta_2^{n+2-j} B^j + \delta_2^2 \sum_{j=0}^{n} a_2^j \ln(1-k) = \\
&= \ln(\gamma^c_{22}) + \sum_{j=0}^{n+2} a_2^j \ln(x) + \sum_{j=1}^{n+2} a_2^j \ln(\varepsilon - \gamma^c_{12} - \gamma^c_{22}) + \delta_2 \ln(\gamma^c_{21}) + \\
&+ \delta_2 \sum_{j=1}^{n+1} a_2^j \ln(\varepsilon - \gamma^c_{11} - \gamma^c_{21}) + \sum_{j=1}^{n} \delta_2^{n+2-j} B^j + \delta_2^2 \sum_{j=0}^{n} a_2^j \ln(1-k) \, .
\end{aligned}
$$

To determine cooperative strategies for this two-step game we solve the following problem

$$\begin{aligned}(H^c_{12}(\gamma^c_{11},\gamma^c_{21},\gamma^c_{12},\gamma^c_{22};x) - V^N_1(x,\delta_1)[n_1-1,n_1])\,\cdot \\ \cdot(H^c_{22}(\gamma^c_{11},\gamma^c_{21},\gamma^c_{12},\gamma^c_{22};x) - \\ -[V^N_2(x,\delta_2)[n_1-1,n_1] + V^{aN}_2(x^{Nn_1},\delta_2)[n_1,n_2]]) = \\ = (H^c_{21} - V^N_1)(H^c_{22} - \tilde{V}^N_2) \to \max_{\gamma^c_{11},\gamma^c_{21},\gamma^c_{12},\gamma^c_{22}},\end{aligned} \tag{18}$$

where $\tilde{V}^N_2$ is the expression in square brackets.

From the first-order conditions we obtain the strategies

$$\gamma^c_{21} = \frac{\varepsilon - \gamma^c_{11}(1+a_1)}{\sum\limits_{j=0}^{n+1} a_2^j},\ \gamma^c_{22} = \frac{\varepsilon - \gamma^c_{12}(1+a_1+a_1^2)}{\sum\limits_{j=0}^{n+2} a_2^j} \tag{19}$$

and the next relation

$$\gamma_{12}^c = \frac{\varepsilon \gamma_{11}^c \sum\limits_{j=1}^{n+2} a_2^j}{\varepsilon a_1 \sum\limits_{j=0}^{n+2} a_2^j + \gamma_{11}^c \left(\sum\limits_{j=1}^{n+2} a_2^j (1 + a_1 + a_1^2) - (a_1 + a_1^2) \sum\limits_{j=0}^{n+2} a_2^j \right)} . \tag{20}$$

So, we express all the parameters using only the first player's strategy on the last step γ_{11}^c, and to determine it we need to solve one of the first-order conditions. Unfortunately it can't be solved analytically, and we give some results of numerical modelling.

The process can be repeated for the n_1-stage game, and we get the next form of the profits

$$H^c_{1n_1}(\gamma^c_{11},\dots,\gamma^c_{1n_1},\gamma^c_{21},\dots,\gamma^c_{2n_1};x) = \\ = \sum_{j=0}^{n_1} a_1^j \ln(x) + \sum_{j=1}^{n_1} \delta_1^{n_1-j} \ln(\gamma^c_{1j}) + \\ + \sum_{j=1}^{n_1} \delta_1^{n_1-j} \sum_{i=1}^{j} a_1^i \ln(\varepsilon - \gamma^c_{1j} - \gamma^c_{2j}) + \delta_1^{n_1} \ln(k)$$

and

$$H^c_{2n_1}(\gamma^c_{11},\dots,\gamma^c_{1n_1},\gamma^c_{21},\dots,\gamma^c_{2n_1};x) = \\ = \sum_{j=0}^{n+n_1} a_2^j \ln(x) + \sum_{j=1}^{n_1} \delta_1^{n_1-j} \ln(\gamma^c_{2j}) + \sum_{j=1}^{n_1} \delta_2^{n_1-j} \sum_{i=1}^{n+j} a_2^i \ln(\varepsilon - \gamma^c_{1j} - \gamma^c_{2j}) + \\ + \sum_{j=1}^{n} \delta_2^{n+n_1-j} B^j + \delta_2^{n_1} \sum_{j=0}^{n} a_2^j \ln(1-k)\,.$$

The cooperative strategies are related as follows

$$\gamma_{1t}^c = \frac{\varepsilon\gamma_{11}^c \sum\limits_{j=t-1}^{n+t} a_2^j}{\varepsilon a_1^{t-1} \sum\limits_{j=0}^{n+t} a_2^j + \gamma_{11}^c \left(\sum\limits_{j=t-1}^{n+t} a_2^j \sum\limits_{j=0}^{t} a_1^j - (a_1^{t-1} + a_1^t) \sum\limits_{j=0}^{n+t} a_2^j \right)},$$

$$\gamma_{2t}^c = \frac{\varepsilon - \gamma_{1t}^c \sum\limits_{j=0}^{t} a_1^j}{\sum\limits_{j=0}^{n+t} a_2^j}.$$

γ_{11}^c can be determined from one of the first-order conditions, for example, the last one

$$a_1^{n-1}(\varepsilon - \gamma_1^1(1 + a_1))(H_1^n - V_1) - a_2^{n-1}(1 + a_2)\gamma_1^1(H_2^n - V_2) = 0\,.$$

Modelling

$$\varepsilon = 0.6\,, \quad \alpha = 0.3\,, \quad n_2 = 20\,, \quad n_1 = 10\,,$$
$$\delta_1 = 0.85\,, \quad \delta_2 = 0.9\,, \quad x_0 = 0.8\,, \quad k = \frac{1}{3}\,.$$

We get $\gamma_{11}^c = 0.2723$. For the first player we compare the cooperative and the noncooperative gains on time interval $[0, n_1]$:

$$V_1^c(x, \delta_1)[0, n_1] = -10.3870 > V_1^N(x, \delta_1)[0, n_1] = -11.9010\,.$$

For the second player we compare the cooperative gain on time interval $[0, n_1]$ plus acting individually on time interval $[n_1, n_2]$ after cooperation, and the noncooperative gain on time interval $[0, n_1]$ plus individual gain on time interval $[n_1, n_2]$ after noncooperation:

$$V_2^c(x,\delta_2)[0,n_1] + V_2^{ac}(x^{cn_1},\delta_2)[n_1,n_2] = -19.6375 >$$
$$> V_2^N(x,\delta_2)[0,n_1] + V_2^{aN}(x^{Nn_1},\delta_2)[n_1,n_2] = -23.2596\,.$$

One can notice that cooperative profits are larger that noncooperative ones for both players.

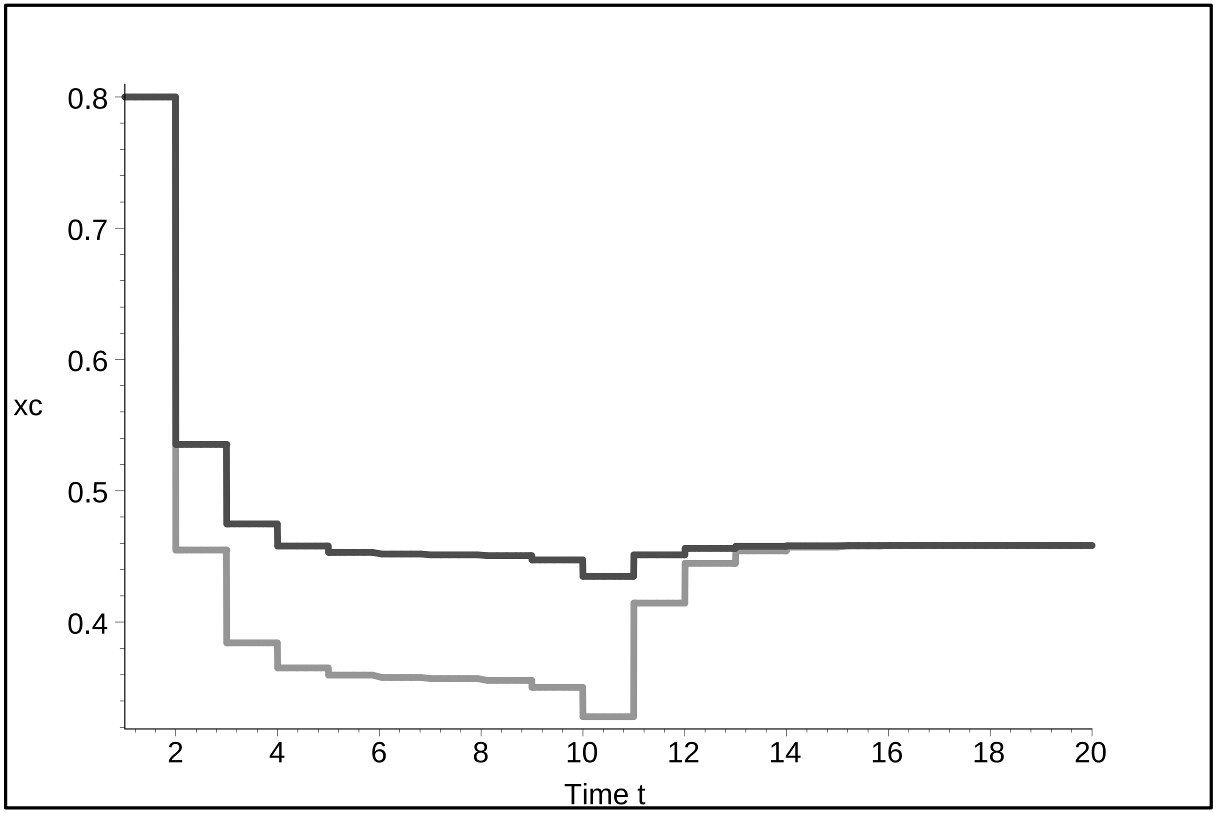

Fig. 6. The population size: dark – cooperative, light – Nash

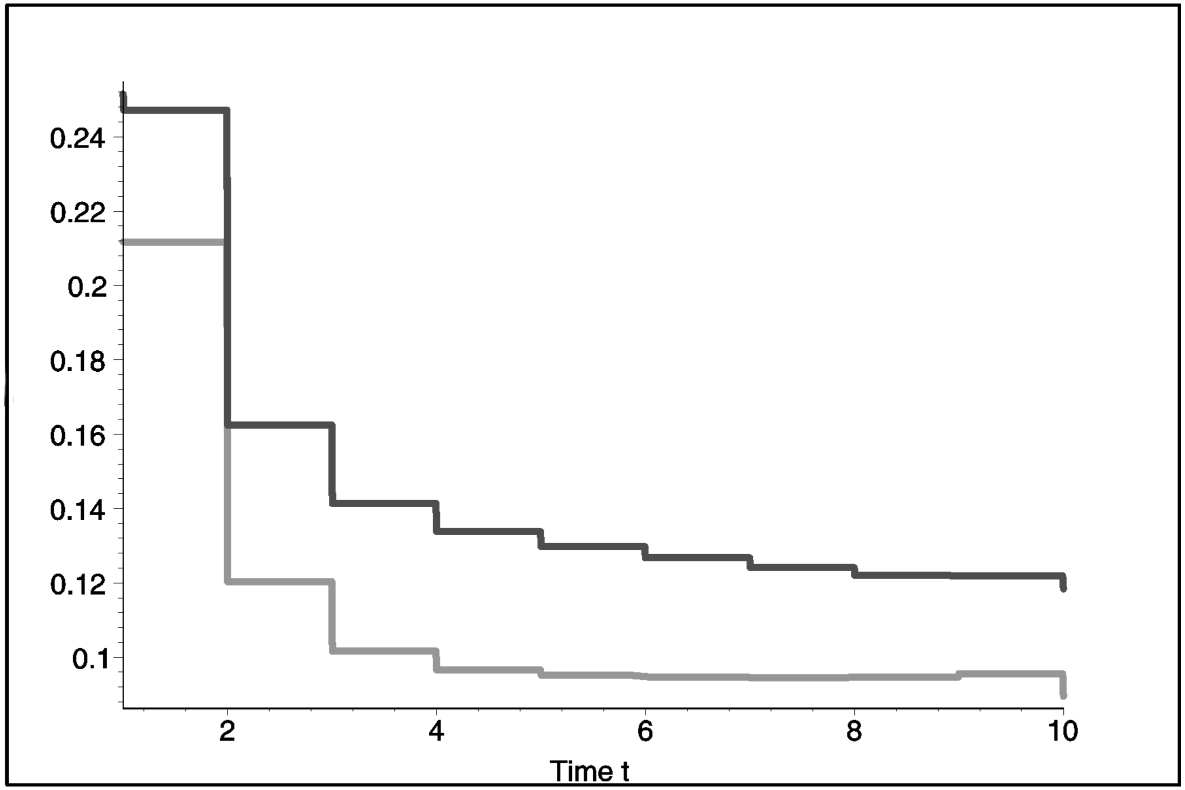

Fig. 7. The catch of player 1: dark – cooperative, light – Nash

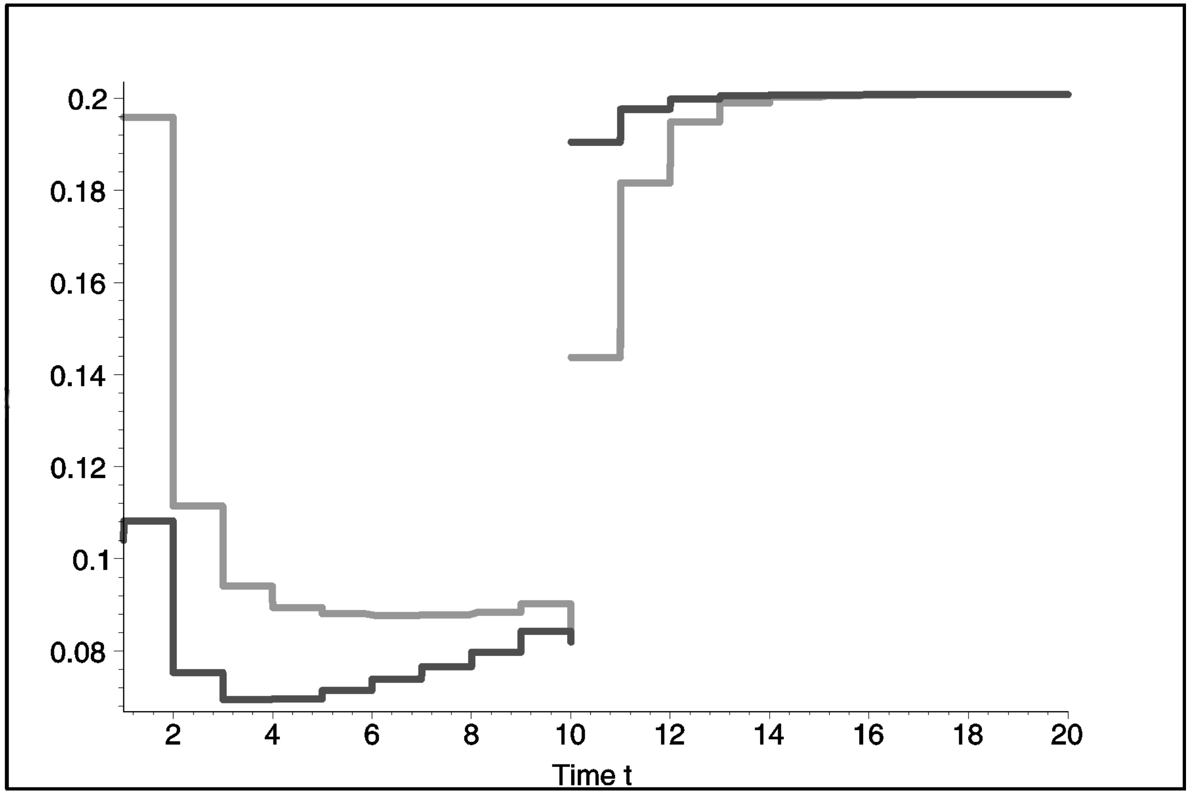

Fig. 8. The catch of player 2: dark – cooperative+individual, light – Nash+individual

Next we compare players' profits for different planning horizons. We show $V_1^c(x, \delta_1)[0, n_1]$ and $V_2^c(x, \delta_2)[0, n_1] + V_2^{ac}(x^{cn_1}, \delta_2)[n_1, n_2]$ for $n_2 = 2 \dots 10$ and $n_1 = 1 \dots n_2 - 1$. One can notice that as n_1 becomes closer to n_2 the difference between players' profits becomes less.

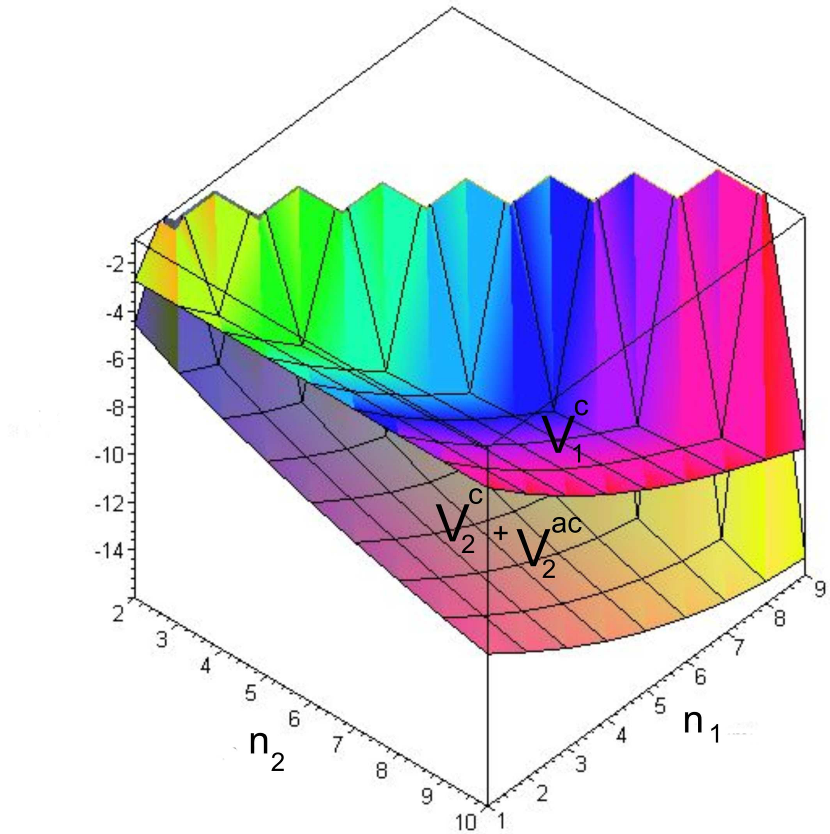

Fig. 9. Players' profits

Also we show that our approach gives a player the payoff that is greater or equal (for some parameters) Nash payoff. So, it again proves that under presented approach it is always profitable to cooperate.

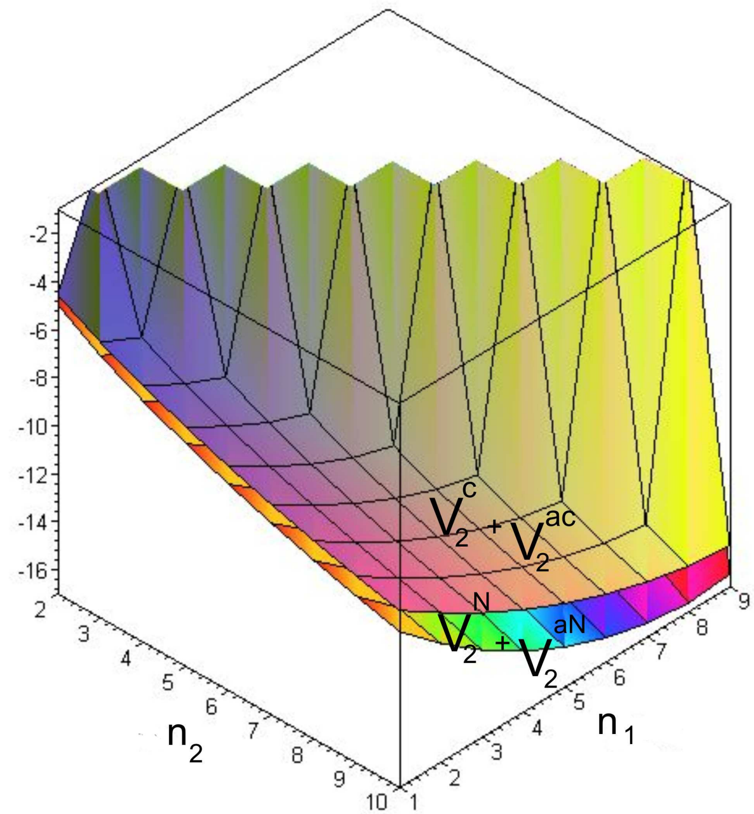

Fig. 10. The second player's profit: Nash and cooperative

5. The model with random harvesting times

The first player harvests the stock for n_1 time moments, and the second – for n_2. n_1 is a random variable with a range $\{1,\dots,n\}$ and corresponding probabilities $\{\theta_1,\dots,\theta_n\}$. n_2 is a random variable with the same range and probabilities $\{\omega_1,\dots,\omega_n\}$.

First, we construct the players' payoffs as mathematical expectations:

$$
\begin{aligned}
H_1 = E\Big\{\sum_{t=1}^{n_1} \delta_1^t \ln(u_{1t}) I_{\{n_1 \le n_2\}} +\\
+\Big(\sum_{t=1}^{n_2} \delta_1^t \ln(u_{1t}) + \sum_{t=n_2+1}^{n_1} \delta_1^t \ln(u_{1t}^a)\Big) I_{\{n_1 > n_2\}}\Big\} =\\
= \sum_{n_1=1}^{n} \theta_{n_1}\Big[\sum_{n_2=n_1}^{n} \omega_{n_2} \sum_{t=1}^{n_1} \delta_1^t \ln(u_{1t}) +\\
+ \sum_{n_2=1}^{n_1-1} \omega_{n_2}\Big(\sum_{t=1}^{n_2} \delta_1^t \ln(u_{1t}) + \sum_{t=n_2+1}^{n_1} \delta_1^t \ln(u_{1t}^a)\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
H_2 = E\Big\{\sum_{t=1}^{n_2} \delta_2^t \ln(u_{2t}) I_{\{n_2 \le n_1\}} + \\
+\Big(\sum_{t=1}^{n_1} \delta_2^t \ln(u_{2t}) + \sum_{t=n_1+1}^{n_2} \delta_2^t \ln(u_{2t}^a)\Big) I_{\{n_2 > n_1\}}\Big\} = \\
= \sum_{n_2=1}^{n} \omega_{n_2}\Big[\sum_{n_1=n_2}^{n} \theta_{n_1} \sum_{t=1}^{n_2} \delta_2^t \ln(u_{2t}) + \\
+ \sum_{n_1=1}^{n_2-1} \theta_{n_1}\Big(\sum_{t=1}^{n_1} \delta_2^t \ln(u_{2t}) + \sum_{t=n_1+1}^{n_2} \delta_2^t \ln(u_{2t}^a)\Big)\Big],
\end{aligned}
$$

where u_{it}^a is a player i's strategy when the opponent quits the game, $i = 1, 2$.

The Nash equilibrium

First we determine the Nash equilibrium as we use it as a status-quo point for the Nash bargaining solution. The value functions for the whole game take the forms

$$V_1^N(1,x) = \max_{u_{11}^N,\ldots,u_{1n}^N} \Big\{ \sum_{n_1=1}^{n} \theta_{n_1} \Big[\sum_{n_2=n_1}^{n} \omega_{n_2} \sum_{t=1}^{n_1} \delta_1^t \ln(u_{1t}^N) + \\ + \sum_{n_2=1}^{n_1-1} \omega_{n_2} \Big(\sum_{t=1}^{n_2} \delta_1^t \ln(u_{1t}^N) + \sum_{t=n_2+1}^{n_1} \delta_1^t \ln(u_{1t}^a) \Big) \Big] \Big\},$$

$$V_2^N(1,x) = \max_{u_{21}^N,\ldots,u_{2n}^N} \Big\{ \sum_{n_2=1}^{n} \omega_{n_2} \Big[\sum_{n_1=n_2}^{n} \theta_{n_1} \sum_{t=1}^{n_2} \delta_2^t \ln(u_{2t}^N) + \\ + \sum_{n_1=1}^{n_2-1} \theta_{n_1} \Big(\sum_{t=1}^{n_1} \delta_2^t \ln(u_{2t}^N) + \sum_{t=n_1+1}^{n_2} \delta_2^t \ln(u_{2t}^a) \Big) \Big] \Big\}.$$

To construct the Bellman equation we define the value functions $V_i^N(\tau, x)$ when the stage τ has arrived:

$$V_1^N(\tau,x) = \max_{u_{1\tau}^N,\dots,u_{1n}^N} \Bigg\{ \sum_{n_1=\tau}^{n} \frac{\theta_{n_1}}{\sum\limits_{l=\tau}^{n} \theta_l} \Bigg[\sum_{n_2=n_1}^{n} \frac{\omega_{n_2}}{\sum\limits_{l=\tau}^{n} \omega_l} \sum_{t=\tau}^{n_1} \delta_1^t \ln(u_{1t}^N) + \\ + \sum_{n_2=\tau}^{n_1-1} \frac{\omega_{n_2}}{\sum\limits_{l=\tau}^{n} \omega_l} \sum_{t=\tau}^{n_2} \delta_1^t \ln(u_{1t}^N) + V_1^a(\tau, n_1) \Bigg] \Bigg\}, \quad (21)$$

$$V_2^N(\tau,x) = \max_{u_{2\tau}^N,\dots,u_{1n}^N} \Bigg\{ \sum_{n_2=\tau}^{n} \frac{\omega_{n_2}}{\sum\limits_{l=\tau}^{n} \omega_l} \Bigg[\sum_{n_1=n_2}^{n} \frac{\theta_{n_1}}{\sum\limits_{l=\tau}^{n} \theta_l} \sum_{t=\tau}^{n_2} \delta_2^t \ln(u_{2t}^N) + \\ + \sum_{n_1=\tau}^{n_2-1} \frac{\theta_{n_1}}{\sum\limits_{l=\tau}^{n} \omega_l} \sum_{t=\tau}^{n_1} \delta_2^t \ln(u_{2t}^N) + V_2^a(\tau, n_2) \Bigg] \Bigg\}, \quad (22)$$

where

$$V_1^a(\tau, n_1) = \sum_{n_2=\tau}^{n_1-1} \frac{\omega_{n_2}}{\sum\limits_{l=\tau}^{n} \omega_l} \sum_{t=n_2+1}^{n_1} \delta_1^t \ln(u_{1t}^a)\,,$$

$$V_2^a(\tau, n_2) = \sum_{n_1=\tau}^{n_2-1} \frac{\theta_{n_1}}{\sum\limits_{l=\tau}^{n} \theta_l} \sum_{t=n_1+1}^{n_2} \delta_2^t \ln(u_{2t}^a)$$

are the profits when player i exploits the stock alone, and they can be estimated easily, as we will show later.

Now we find the relation between $V_i^N(\tau, x)$ and $V_i^N(\tau+1, x)$. From (21) we get

$$V_1^N(\tau, x) = \delta_1^\tau \ln(u_{1\tau}^N) + P_\tau^{\tau+1} V_1^N(\tau+1, x) + C_{1\tau} \sum_{n_1=\tau+1}^{n} \theta_{n_1} \sum_{t=\tau}^{n_1} \delta_1^t \ln(u_{1t}^a)\,, \tag{23}$$

$$V_2^N(\tau,x) = \delta_2^\tau \ln(u_{2\tau}^N) + P_\tau^{\tau+1} V_2^N(\tau+1,x) + C_{2\tau} \sum_{n_2=\tau+1}^{n} \omega_{n_2} \sum_{t=\tau}^{n_2} \delta_2^t \ln(u_{2t}^a)\,, \tag{24}$$

where

$$P_\tau^{\tau+1} = \frac{\sum\limits_{l=\tau+1}^{n} \omega_l}{\sum\limits_{l=\tau}^{n} \omega_l} \frac{\sum\limits_{l=\tau+1}^{n} \theta_l}{\sum\limits_{l=\tau}^{n} \theta_l}\,, C_{1\tau} = \frac{\omega_\tau}{\sum\limits_{l=\tau}^{n} \omega_l} \frac{1}{\sum\limits_{l=\tau}^{n} \theta_l}\,,\ C_{2\tau} = \frac{\theta_\tau}{\sum\limits_{l=\tau}^{n} \theta_l} \frac{1}{\sum\limits_{l=\tau}^{n} \omega_l}\,.$$

From the previous case with fixed harvesting times (see (17)), where we maximized the second player's profit when acting individually on time interval $[n_1, n_2]$) we can get

$$u_{it}^a = \frac{\varepsilon(1-a_i)}{1-a_i^t} x\,,$$

$$\sum_{t=\tau}^{n_i} \delta_i^t \ln(u_{it}^a) = \sum_{j=0}^{n_i-\tau} a_i^j \ln x + \sum_{j=1}^{n_i-\tau} \delta_i^{n_i-\tau-j} D_i^j \,, \tag{25}$$

where

$$D_i^j = \sum_{l=0}^{j} a_i^l \ln\Big(\frac{\varepsilon}{\sum_{p=0}^{j} a_i^p}\Big) + \sum_{l=1}^{j} a_i^l \ln(\sum_{p=1}^{j} a_i^p)\,.$$

As usual for "fish war" models we seek the value functions in the form $V_i^N(\tau, x) = A_i^\tau \ln x + B_i^\tau$ and the Nash strategies in linear form $u_{i\tau}^N = \gamma_{i\tau}^N x$, $i = 1, 2$.

From the first-order conditions we get the Nash strategies

$$\gamma_{1\tau}^N = \frac{\varepsilon \delta_1^\tau A_2^\tau}{\delta_1^\tau A_2^\tau + \delta_2^\tau A_1^\tau + \alpha A_1^\tau A_2^\tau P_\tau^{\tau+1}}\,,\ \gamma_{2\tau}^N = \frac{\varepsilon \delta_2^\tau A_1^\tau}{\delta_1^\tau A_2^\tau + \delta_2^\tau A_1^\tau + \alpha A_1^\tau A_2^\tau P_\tau^{\tau+1}}\,.$$

Coefficients A_i^τ and B_i^τ are derived from (21) and (22)

$$A_1^\tau = \frac{\delta_1^\tau + C_{1\tau} \sum\limits_{n_1=\tau+1}^{n} \theta_{n_1} \sum\limits_{j=0}^{n_1-\tau} a_1^j}{1-\alpha P_\tau^{\tau+1}}, \; A_2^\tau = \frac{\delta_2^\tau + C_{2\tau} \sum\limits_{n_2=\tau+1}^{n} \omega_{n_2} \sum\limits_{j=0}^{n_2-\tau} a_2^j}{1-\alpha P_\tau^{\tau+1}}, \tag{26}$$

$$B_1^\tau = \frac{\delta_1^\tau \ln(\gamma_{1\tau}^N) + \alpha A_1^\tau P_\tau^{\tau+1} \ln(\varepsilon - \gamma_{1\tau}^N - \gamma_{2\tau}^N) + C_{1\tau} \sum\limits_{n_1=\tau+1}^{n} \theta_{n_1} \sum\limits_{j=1}^{n_1-\tau} \delta_1^{n_1-\tau-j} D_1^j}{1-P_\tau^{\tau+1}},$$

$$B_2^\tau = \frac{\delta_2^\tau \ln(\gamma_{2\tau}^N) + \alpha A_2^\tau P_\tau^{\tau+1} \ln(\varepsilon - \gamma_{1\tau}^N - \gamma_{2\tau}^N) + C_{2\tau} \sum\limits_{n_2=\tau+1}^{n} \omega_{n_2} \sum\limits_{j=1}^{n_2-\tau} \delta_2^{n_2-\tau-j} D_2^j}{1-P_\tau^{\tau+1}}.$$

So, we determined the Nash strategies and the Nash payoffs $V_i^N(\tau,x) = A_i^\tau \ln x + B_i^\tau$, $i=1,2$.

The cooperative behavior

We construct cooperative strategies and the payoff maximizing the Nash product for the whole game, so we need to solve the following problem

$$
\begin{aligned}
&(V_1^c(1,x) - V_1^N(1,x))(V_2^c(1,x) - V_2^N(1,x)) = \\
&= \Big(\sum_{n_1=1}^{n} \theta_{n_1}\Big[\sum_{n_2=n_1}^{n} \omega_{n_2} \sum_{t=1}^{n_1} \delta_1^t \ln(u_{1t}^c) + \\
&+ \sum_{n_2=1}^{n_1-1} \omega_{n_2}\Big(\sum_{t=1}^{n_2} \delta_1^t \ln(u_{1t}^c) + \sum_{t=n_2+1}^{n_1} \delta_1^t \ln(u_{1t}^a)\Big)\Big] - V_1^N(1,x)\Big) \cdot \\
&\cdot\Big(\sum_{n_2=1}^{n} \omega_{n_2}\Big[\sum_{n_1=n_2}^{n} \theta_{n_1} \sum_{t=1}^{n_2} \delta_2^t \ln(u_{2t}^c) + \\
&+ \sum_{n_1=1}^{n_2-1} \theta_{n_1}\Big(\sum_{t=1}^{n_1} \delta_2^t \ln(u_{2t}^c) + \sum_{t=n_1+1}^{n_2} \delta_2^t \ln(u_{2t}^a)\Big)\Big] - V_2^N(1,x)\Big) \to \max, \qquad (27)
\end{aligned}
$$

where $V_i^N(1, x) = A_i^N \ln x + B_i^N$, $i = 1, 2$ are the non-cooperative gains determined in (26).

When step τ has arrived we determine the cooperative value functions $V_i^c(\tau, x)$ as

$$V_1^c(\tau,x) = \max_{u_{1\tau}^c,\ldots,u_{1n}^c} \Big\{ \sum_{n_1=\tau}^{n} \frac{\theta_{n_1}}{\sum\limits_{l=\tau}^{n} \theta_l} \Big[\sum_{n_2=n_1}^{n} \frac{\omega_{n_2}}{\sum\limits_{l=\tau}^{n} \omega_l} \sum_{t=\tau}^{n_1} \delta_1^t \ln(u_{1t}^c) + \\ + \sum_{n_2=\tau}^{n_1-1} \frac{\omega_{n_2}}{\sum\limits_{l=\tau}^{n} \omega_l} \sum_{t=\tau}^{n_2} \delta_1^t \ln(u_{1t}^c) + V_1^a(\tau, n_1) \Big] \Big\}, \quad (28)$$

$$V_2^c(\tau,x) = \max_{u_{2\tau}^c,\ldots,u_{2n}^c} \Big\{ \sum_{n_2=\tau}^{n} \frac{\omega_{n_2}}{\sum\limits_{l=\tau}^{n} \omega_l} \Big[\sum_{n_1=n_2}^{n} \frac{\theta_{n_1}}{\sum\limits_{l=\tau}^{n} \theta_l} \sum_{t=\tau}^{n_2} \delta_2^t \ln(u_{2t}^c) + \\ + \sum_{n_1=\tau}^{n_2-1} \frac{\theta_{n_1}}{\sum\limits_{l=\tau}^{n} \omega_l} \sum_{t=\tau}^{n_1} \delta_2^t \ln(u_{2t}^c) + V_2^a(\tau, n_2) \Big] \Big\}. \quad (29)$$

Similarly to the Nash payoffs we get the relation between the cooperative payoffs at time moments τ and $\tau+1$:

$$V_1^c(\tau,x) = \delta_1^\tau \ln(u_{1\tau}^c) + P_\tau^{\tau+1} V_1^c(\tau+1,x) + C_{1\tau} \sum_{n_1=\tau+1}^{n} \theta_{n_1} \sum_{t=\tau}^{n_1} \delta_1^t \ln(u_{1t}^a)\,,$$

$$V_2^c(\tau,x) = \delta_2^\tau \ln(u_{2\tau}^c) + P_\tau^{\tau+1} V_2^c(\tau+1,x) + C_{2\tau} \sum_{n_2=\tau+1}^{n} \omega_{n_2} \sum_{t=\tau}^{n_2} \delta_2^t \ln(u_{2t}^a)\,.$$

We start when step n has arrived. Since on the next step $n+1$ the payoffs of both players are equal to zero, the optimal strategies are Nash equilibrium strategies and

$$V_i^c(n,x) = \delta_i^n \ln(u_{in}^c) = V_i^N(n,x) = \delta_i^n \ln(\gamma_{in}^N x) = A_i \ln x + B_i\,,\ i=1,2\,, \tag{30}$$

where

$$A_i = \delta_i^n\,,\ B_i = \delta_i^n \ln(\gamma_{1n}^N) = \delta_i^n \ln(\frac{\varepsilon}{2})\,,\ i=1,2\,.$$

Now we suppose that step $n-1$ has arrived. We have problem (27) in the form

$$(V_1^c(n-1,x) - V_1^N(n-1,x))(V_2^c(n-1,x) - V_2^N(n-1,x)) \to \max\,, \tag{31}$$

where

$$V_1^c(n-1,x) = \delta_1^{n-1} \ln(u_{1n-1}^c) + P_{n-1}^n V_1^c(n, (\varepsilon x - u_{1n-1}^c - u_{2n-1}^c)^\alpha) + \\ + C_{1n-1}\theta_n \sum_{t=n-1}^{n} \delta_1^t \ln(u_{1t}^a)\,,$$

$$V_2^c(n-1,x) = \delta_2^{n-1} \ln(u_{2n-1}^c) + P_{n-1}^n V_2^c(n, (\varepsilon x - u_{1n-1}^c - u_{2n-1}^c)^\alpha) + \\ + C_{2n-1}\omega_n \sum_{t=n-1}^{n} \delta_2^t \ln(u_{2t}^a)\,.$$

As usual we seek the strategies in the linear form $u_{in-1}^c = \gamma_{in-1}^c x$. From first-order conditions we get

$$\gamma_{2n-1}^c = \frac{\delta_1^{n-1}\delta_2^{n-1}\varepsilon - \delta_2^{n-1}\gamma_{1n-1}^c(\delta_1^{n-1} + P_{n-1}^n \alpha A_1)}{\delta_1^{n-1}(\delta_2^{n-1} + P_{n-1}^n \alpha A_2)} . \tag{32}$$

Now we pass to the situation when step $n-2$ has arrived. We have problem (27) in the form

$$(V_1^c(n-2,x) - V_1^N(n-2,x))(V_2^c(n-2,x) - V_2^N(n-2,x)) \to \max , \tag{33}$$

where

$$V_1^c(n-2,x) = \delta_1^{n-2}\ln(u_{1n-2}^c) +$$

$$+P_{n-2}^{n-1}V_1^c(n-1,(\varepsilon x - u_{1n-2}^c - u_{2n-2}^c)^\alpha) + C_{1n-2}\sum_{n_1=n-1}^{n}\theta_{n_1}\sum_{t=n-2}^{n_1}\delta_1^t\ln(u_{1t}^a),$$

$$V_2^c(n-2,x) = \delta_2^{n-2}\ln(u_{2n-2}^c) +$$

$$+P_{n-2}^{n-1}V_2^c(n-1,(\varepsilon x - u_{1n-2}^c - u_{2n-2}^c)^\alpha) + C_{2n-2}\sum_{n_2=n-1}^{n}\omega_{n_2}\sum_{t=n-2}^{n_2}\delta_2^t\ln(u_{2t}^a).$$

We seek the strategies in the linear form $u_{in-2}^c = \gamma_{in-2}^c x$. From the first-order conditions we get

$$\gamma_{2n-2}^c = \frac{\delta_1^{n-2}\delta_2^{n-2}\varepsilon - \delta_2^{n-2}\gamma_{1n-2}^c(\delta_1^{n-2} + \alpha\delta_1^{n-1}P_{n-2}^{n-1} + \alpha^2 A_1 P_{n-2}^{n-1}P_{n-1}^n)}{\delta_1^{n-2}(\delta_2^{n-2} + \alpha\delta_2^{n-1}P_{n-2}^{n-1} + \alpha^2 A_2 P_{n-2}^{n-1}P_{n-1}^n)}. \tag{34}$$

Let's denote

$$G_1^1 = \delta_1^{n-1} + P_{n-1}^n \alpha A_1\,,\ G_1^2 = \delta_2^{n-1} + P_{n-1}^n \alpha A_1\,,$$
$$G_2^1 = \delta_1^{n-2} + \alpha\delta_1^{n-1} P_{n-2}^{n-1} + \alpha^2 A_1 P_{n-2}^{n-1} P_{n-1}^n\,,$$
$$G_2^2 = \delta_2^{n-2} + \alpha\delta_2^{n-1} P_{n-2}^{n-1} + \alpha^2 A_2 P_{n-2}^{n-1} P_{n-1}^n\,.$$

Then (32) and (34) take the forms

$$\gamma_{2n-1}^c = \frac{\delta_1^{n-1}\delta_2^{n-1}\varepsilon - \delta_2^{n-1}\gamma_{1n-1}^c G_1^1}{\delta_1^{n-1} G_1^2}\,,\ \gamma_{2n-2}^c = \frac{\delta_1^{n-2}\delta_2^{n-2}\varepsilon - \delta_2^{n-2}\gamma_{1n-2}^c G_2^1}{\delta_1^{n-2} G_2^2}\,.$$

And we can express γ_{1n-2}^c with γ_{1n-1}^c:

$$\gamma_{1n-2}^c = \delta_1^{n-2}\varepsilon \frac{\gamma_{1n-1}^c G_1^2}{\delta_1^{n-1}\varepsilon G_2^2 + \gamma_{1n-1}^c (G_2^1 G_1^2 - G_1^1 G_2^2)}\,.$$

The value functions take the forms

$$
\begin{aligned}
V_1^c(n-2,x) &= \delta_1^{n-2}\ln(u_{1n-2}^c) + \alpha P_{n-2}^{n-1} G_1^1 \ln(\varepsilon x - u_{1n-2}^c - u_{2n-2}^c) + \\
&+ P_{n-2}^{n-1}[\delta_1^{n-1}\ln(\gamma_{1n-1}^c) + P_{n-1}^n \alpha A_1 \ln(\varepsilon - \gamma_{1n-1}^c - \gamma_{2n-1}^c) + P_{n-1}^n B_1] + \\
&+ P_{n-2}^{n-1} C_{1n-1}\theta_n \sum_{t=n-1}^{n} \delta_1^t \ln(u_{1t}^a) + C_{1n-2} \sum_{n_1=n-1}^{n} \theta_{n_1} \sum_{t=n-2}^{n_1} \delta_1^t \ln(u_{1t}^a)\,, \\
V_2^c(n-2,x) &= \delta_2^{n-2}\ln(u_{2n-2}^c) + \alpha P_{n-2}^{n-1} G_1^2 \ln(\varepsilon x - u_{1n-2}^c - u_{2n-2}^c) + \\
&+ P_{n-2}^{n-1}[\delta_2^{n-1}\ln(\gamma_{2n-1}^c) + P_{n-1}^n \alpha A_2 \ln(\varepsilon - \gamma_{1n-1}^c - \gamma_{2n-1}^c) + P_{n-1}^n B_2] + \\
&+ P_{n-2}^{n-1} C_{2n-1}\omega_n \sum_{t=n-1}^{n} \delta_2^t \ln(u_{2t}^a) + C_{2n-2} \sum_{n_2=n-1}^{n} \omega_{n_2} \sum_{t=n-2}^{n_2} \delta_2^t \ln(u_{2t}^a)\,.
\end{aligned}
$$

Continuing the process for k steps we get the payoffs in the form

$$
\begin{aligned}
&V_i^c(n-k,x) = \\
&= \delta_i^{n-k} \ln(u_{in-k}^c) + \alpha P_{n-k}^{n-k+1} G_{n-k+1}^i \ln(\varepsilon x - u_{1n-k}^c - u_{2n-k}^c) + \\
&+ \sum_{l=2}^{k-1} P_{n-k}^{n-l} [\delta_i^{n-l} \ln(\gamma_{in-l}^c) + \alpha P_{n-l}^{n-l+1} \ln(\varepsilon - \gamma_{1n-l}^c - \gamma_{2n-l}^c)] + \\
&+ P_{n-k}^{n-1} [\delta_i^{n-1} \ln(\gamma_{in-1}^c) + P_{n-1}^n \alpha A_i \ln(\varepsilon - \gamma_{1n-1}^c - \gamma_{2n-1}^c) + P_{n-1}^n B_i] + \\
&+ \sum_{l=1}^{k} P_{n-k}^{n-l} C_{in-l} V_i^l(n_i), \qquad (35)
\end{aligned}
$$

where

$$V_1^l(n_1) = \sum_{n_1=n-l+1}^{n} \theta_{n_1} \sum_{t=n-l}^{n_1} \delta_1^t \ln(u_{1t}^a)\,,$$

$$V_2^l(n_2) = \sum_{n_2=n-l+1}^{n} \omega_{n_2} \sum_{t=n-l}^{n_2} \delta_2^t \ln(u_{2t}^a)\,,$$

$$G_k^1 = \sum_{l=1}^{k} \delta_1^{n-l} \alpha^{k-l} P_{n-k}^{n-l} + \alpha^k A_1 P_{n-k}^n\,,$$

$$G_k^2 = \sum_{l=1}^{k} \delta_2^{n-l} \alpha^{k-l} P_{n-k}^{n-l} + \alpha^k A_2 P_{n-k}^n\,.$$

The cooperative strategies are related as follows

$$\gamma^c_{2n-k} = \frac{\delta_1^{n-k}\delta_2^{n-k}\varepsilon - \delta_2^{n-k}\gamma^c_{1n-k}G^1_k}{\delta_1^{n-k}G^2_k}, \tag{36}$$

$$\gamma^c_{1n-k} = \frac{\delta_1^{n-k}\varepsilon\gamma^c_{1n-1}G^2_1}{\delta_1^{n-1}\varepsilon G^2_k + \gamma^c_{1n-1}(G^1_kG^2_1 - G^1_1G^2_k)}. \tag{37}$$

And γ^c_{1n-1} can be determined from one of the first-order conditions.

Modelling

We use Monte-Carlo method for $n = 10$.

For the same parameters and the next probabilities

$$\theta_i = 0.1\,,\ \omega_i = 0.005i + 0.0725$$

we get the expected cooperative and Nash payoffs

$$V_1^c(1,x) = -6.2151 > V_1^N(1,x) = -10.1958\,,$$

$$V_2^c(1,x) = -7.3256 > V_2^N(1,x) = -12.8829\,.$$

Fig. 6 presents the results of the modelling with 50 simulations for the Nash equilibrium, Fig. 7 – for the cooperative equilibrium. Points are the results of simulations and circles denote the expected payoffs determined in (26) and (35).

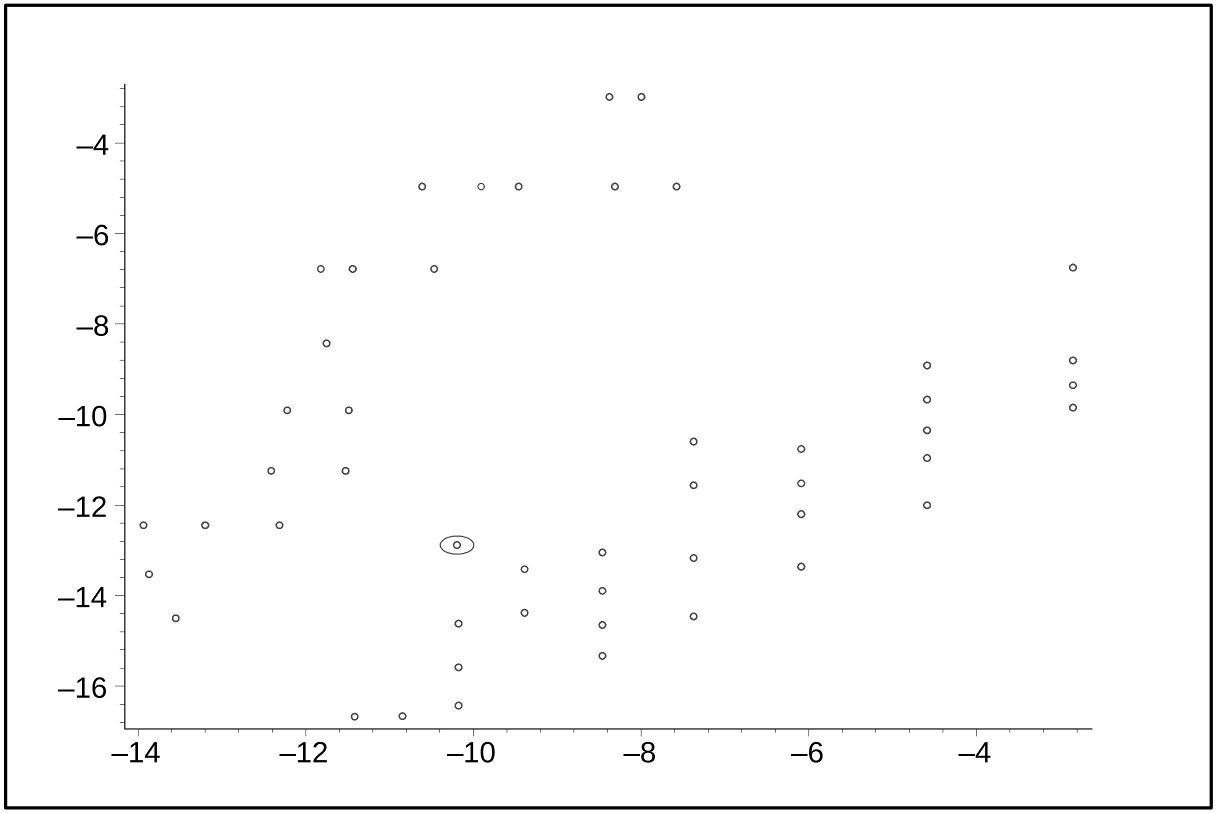

Fig. 11. Nash equilibrium

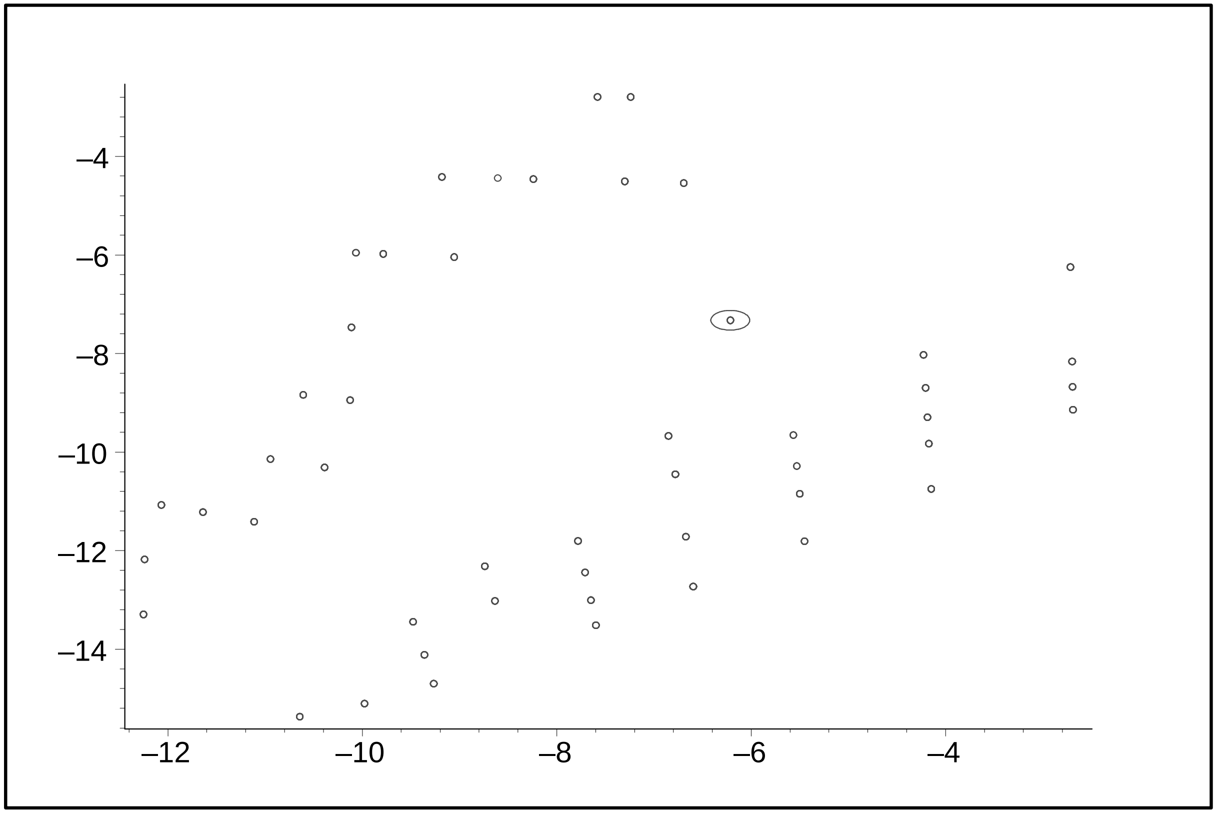

Fig. 12. Cooperative equilibrium

We considered a discrete time bioresource management problem with two players which differ not only in discount factors, but in harvesting times. How to determine the cooperative gains in both cases has not been studied yet.

In the first model, participation planning horizons are known. Here one player leaves the game at a fixed time moment and receives a portion of the remaining stock as compensation. The second player continues exploitation until the end of the game individually. To construct the cooperative strategies we used the Nash bargaining scheme for the whole planning horizon.

In the second model, the harvesting times are random variables and the distribution functions for the players' planning horizons differ. First, we constructed the Nash equilibrium and used it as a status-quo point. Second, we determined the cooperative strategies using the Nash bargaining procedure.

References

1. Breton M., Keoula M.Y. (2014) 'A great fish war model with asymmetric players' // Ecological Economics **97**: 209–223..

2. Fisher R.D., Mirman L.J. (1996), 'The complete fish wars: biological and dynamic interactions', *J. of Environmental Economics and Management* **30**: 34–42.

3. Levhari D., Mirman L.J. (1980), 'The great fish war: an example using a dynamic Cournot-Nash solution', *The Bell Journal of Economic* **11**, N 1 : 322–334.

5. Mazalov V.V., Rettieva A.N. (2010), 'Fish wars and cooperation maintenance', *Ecological Modelling* **221**: 1545–1553.

6. Rettieva A.N. 'Bioresource management problem with assymetric players', *Mathematical Game Theory and Applications* **4**, N 4: (in Russian).