Several Necessary Conditions For Uniformity of Finite Systems of Many-valued Logic

Pavel Tarasov

Lomonosov Moscow State University,
Faculty of Mechanics and Mathematics,
Department of Discrete Mathematics

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Definitions

$$E_k = \{0, \ldots, k-1\}$$

$$E_k^n = \underbrace{E_k \times E_k \ldots \times \ldots E_k}_{\text{n times}}$$

P_k — set of all functions of type $f : E_k^n \to E_k$

$P_{k,s}$ — set of all functions of type $f : E_k^n \to E_s$

Definitions

Formula is called trivial, if it is a symbol of a variable

$A \subseteq P_k$, $[A]$ — set of all functions from P_k, which can be realized by non-trivial formula over A

Definitions

$L(\Phi)$ — complexity of formula Φ (number of symbols of variables occurring in Φ)

$D(\Phi)$ — depth of formula Φ

$A \subseteq P_k$, $f \in [A]$,

$D(f) = \min D(\Phi)$, $L(f) = \min L(\Phi)$
where min is taken over all formulas over A realizing f

Definitions

For arbitrary finite system

$$A = \{f_1(x_1, \ldots, x_{n_1}), \ldots, f_m(x_1, \ldots, x_{n_m})\} \subseteq P_k,$$

the following inequalities holds

$$L_A(f) \le n^{D_A(f)} \quad \text{and} \quad \log_n L_A(f) \le D_A(f),$$

where $n = \max\limits_{i=1..m} n_i$

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Uniformity

The system $A \subset P_k$, $|A| < \infty$, is called a uniform system if $\exists\ c, d$ s.t. $\forall\ f \in [A]$

$$D_A(f) \leq c \log_2 L_A(f) + d$$

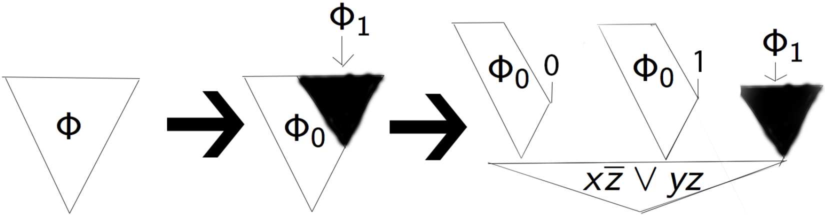
History

1968 1971	Khrapchenko Spira	All finite complete systems of Boolean functions are uniform.
1983	Wegener	All finite systems, generating the class of all monotone Boolean functions are uniform.
1985 1987	Ugolnikov Ragaz	All finite systems of Boolean functions are uniform. Example of non-uniform system from $P_{3,2}$.
2000	Safin Dudakova Dudorov	All finite systems, generating several maximal classes from P_k are uniform.

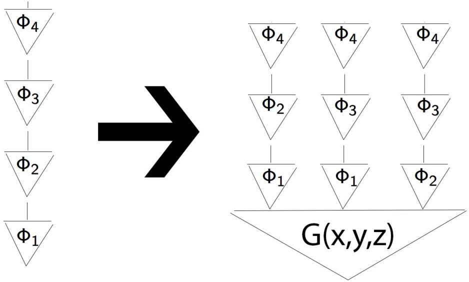
History

Main idea of most proofs:

1. split 'complex' formula into two complex subformulas
2. using special function construct equal formula from subformulas



Generalization:split into more subformulas and use more complex function



The main problem: find function G.

Function $f(x_1, \ldots, x_n) \in P_k$ is called a near unanimity function, if for arbitrary α, $\beta \in E_k$ equalities holds

$$f(\beta, \alpha, \ldots, \alpha) = f(\alpha, \beta, \alpha, \ldots, \alpha) = \ldots = f(\alpha, \ldots, \alpha, \beta) = \alpha.$$

Projection

$f(x_1, \ldots, x_n) \in P_{k,s}, \quad g(x_1, \ldots, x_n) \in P_s.$
$g = \mathrm{pr}_s f$ ("projection" on set P_s) if $\forall \widetilde{\alpha} \in E_s^n : f(\widetilde{\alpha}) = g(\widetilde{\alpha})$.

Theorem (Tarasov, 2013)

All finite systems $A \subset P_{k,s}$, such that $[\mathrm{pr}_s A]$ contains near unanimity function are uniform.

Let K, D and L denote closed classes of Boolean conjunctions, disjunctions and linear functions respectively.

Denote

$O^\infty = \{f(x_1,\ldots,x_n) \in P_2 \mid \exists i : f(x_1,\ldots,x_n) \geq x_i\};$

$I^\infty = \{f(x_1,\ldots,x_n) \in P_2 \mid \exists i : f(x_1,\ldots,x_n) \leq x_i\}.$

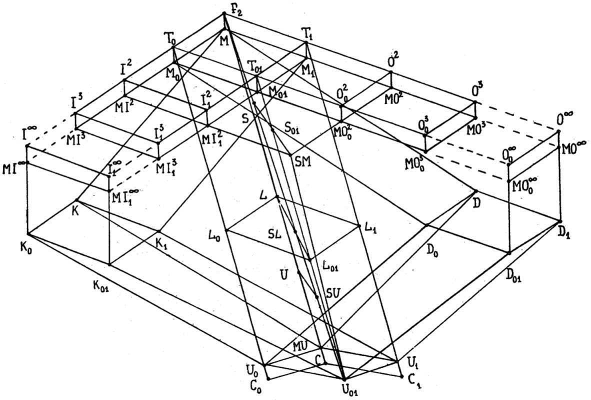
Corollary

All finite systems $A \subseteq P_{k,2}$ such that

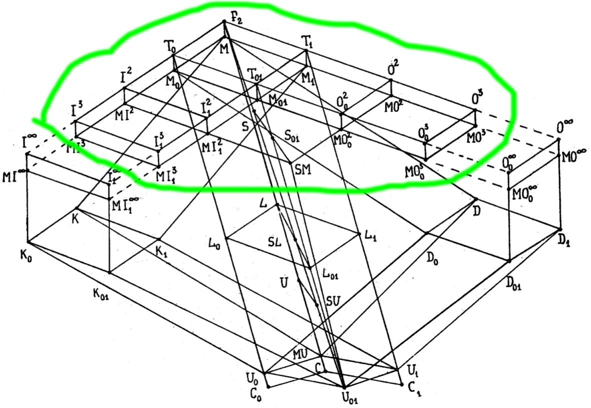
$$\mathrm{pr}_2 A \not\subset O^\infty \cup I^\infty \cup D \cup K \cup L.$$

are uniform.

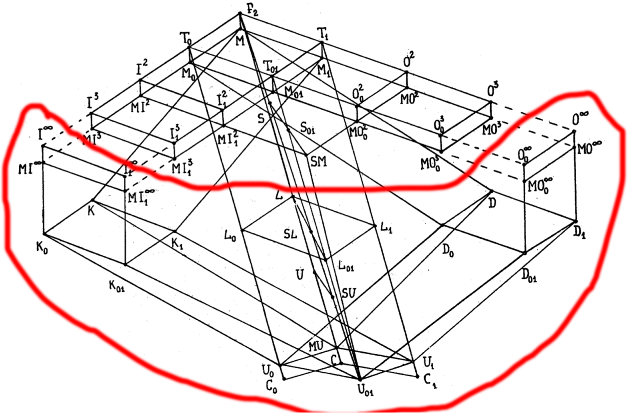
This is the well-known Post lattice of closed classes of P_2



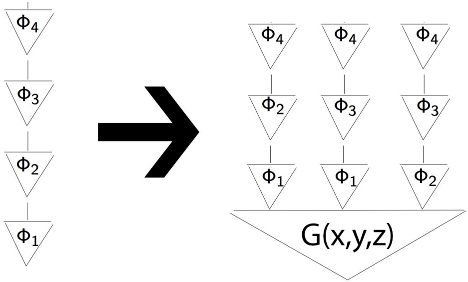
From corollary we get, that all classes from $P_{k,2}$, which are projecting into classes in green area are uniform.



But for classes in the red area it's easy to construct an example of non-uniform system projecting into them. We will strict our attention on them.



Classes from $P_{k,2}$



We can construct function G using near-unanimity functions.
But in red classes there are no near-unanimity function.

Example of uniform system

Example in $P_{3,2}$: $A = \{f(x, y_1, y_2, y_3)\}$

$$f(\alpha, \beta_1, \beta_2, \beta_3) = \begin{cases} 0, & \text{if } \alpha \in E_2 \text{ or } \beta_i \notin E_2; \\ y_1 y_2 \vee y_1 y_3 \vee y_2 y_3, & \text{if } \alpha = 2 \text{ and } \beta_i \in E_2. \end{cases}$$

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If formula Φ over A realizes function g, s.t. $g \neq 0$, then Φ is a formula of a form

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If formula Φ over A realizes function g, s.t. $g \neq 0$, then Φ is a formula of a form

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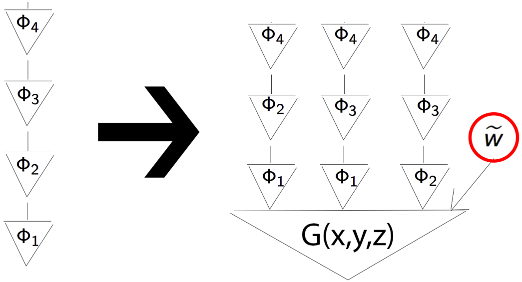
Thus, we can always use the function $y_1 y_2 \vee y_1 y_3 \vee y_2 y_3$, which is near-unanimity function!

Thus, we will modify a construction a little bit.



We add some variables $\widetilde{w}$ to function g, and this allow us to use 'good' functions, even if set $\mathrm{pr}A$ do not contain any good functions.

Example of uniform system

What other 'good' functions can be used to prove uniformity?

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What other 'good' functions can be used to prove uniformity?

Konjunctions and disjunctions!

For example, the system of Boolean functions $\{xy\}$ is uniform, but the set $[\{xy\}]$ does not contain near-unanimity function.

Definitions

Def: partial order on the set E_k : $1 > 0$, all other elements are incomparable.

$\widetilde{\alpha} = (\alpha_1, \ldots, \alpha_n), \widetilde{\beta} = (\beta_1, \ldots, \beta_n) \in E_k^n$.
$\widetilde{\alpha} \geq \widetilde{\beta}$ if $\alpha_i \geq \beta_i$ for all $i \in 1..n$.

Function $f(x_1, \ldots, x_n) \in P_{k,2}$ is monotone, if $\forall \widetilde{\alpha}, \widetilde{\beta} \in E_k^n$, s.t. $\widetilde{\alpha} \geq \widetilde{\beta}$ we have $f(\widetilde{\alpha}) \geq f(\widetilde{\beta})$.

Let $f(x_1, \ldots, x_n) \in P_{k,2}$, $i \in \{1, \ldots, n\}$. Define

$$M_f^{x_i} = \bigcup_{\widetilde{\alpha} \in E_k^{n-1}} \{\mathrm{pr}_2 f(\alpha_1, \ldots, \alpha_{i-1}, x, \alpha_i, \ldots, \alpha_{n-1})\},$$

$$V_f^{x_i} = \{\widetilde{\alpha} \in E_k^{n-1} \mid \mathrm{pr}_2 f(\alpha_1, \ldots, \alpha_{i-1}, x, \alpha_i, \ldots, \alpha_{n-1}) = x\}.$$

Definitons

Finite $A \subset P_{k,2}$ has property $\#$ if $\exists q \geq 3$ s.t. $\forall f(x_1, \ldots, x_n) \in A$, $\forall i \in \{1, \ldots, n\}$ $\exists g(x_1, \ldots, x_{n-1}, y_1, \ldots, y_q) \in [A]$ s.t. $\forall \widetilde{\alpha} \in V_f^{x_i}$

1. if $M_f^{x_i} = \{0, 1, x\}$, then function $\mathrm{pr}_2 g(\widetilde{\alpha}, \widetilde{y})$ is a near-unanimity function;
2. if $M_f^{x_i} = \{0, x\}$, then $\mathrm{pr}_2 g(\widetilde{\alpha}, \widetilde{y}) \in M_{01} \setminus O^{\infty}$;
3. if $M_f^{x_i} = \{1, x\}$, then $\mathrm{pr}_2 g(\widetilde{\alpha}, \widetilde{y}) \in M_{01} \setminus I^{\infty}$.

Theorem 1

Finite system of monotone functions from $P_{k,2}$ is uniform only if it has property $\#$.

If system $A \subset P_{k,2}$ does not have property $\#$, there is a constant $c > 0$ and sequence of functions $\{f_1, f_2, \ldots\} \in [A]$, such that

$$L(f_i) > i \quad \text{and} \quad D(f_i) > cL(f_i)$$

Theorem 2

Let A be a finite system of monotone function from $P_{k,2}$ and A has property $\#$. Than there exist such constants c and d that for any function $f \in [A]$ the inequality $D_A(f) \leq c \log_2^2 L_A(f) + d$ holds.

Thus, we haven't prove that all such systems are uniform, but there is a good upper bound for depth of function in such systems.

Results

Theorem 3

It's possible to check if a system has property # in finite time.

Thank you!