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ODDS THEOREM WITH MULTIPLE SELECTION CHANCES

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Abstract

We study the multi-selection version of the so-called *odds theorem* by Bruss [3]. We observe a finite number of independent 0/1 (failure/success) random variables sequentially and want to select the last success. We derive the optimal selection rule when m ($\geq$ 1) selection chances are given and find that the optimal rule has the form of a combination of multiple odds-sums. We provide a formula for computing the maximum probability of selecting the last success when we have m selection chances and also provide closed-form formulas for $m = 2$ and 3. For $m = 2$, we further give the bounds for the maximum probability of selecting the last success and derive its limit as the number of observations goes to infinity. An interesting implication of our result is that the limit of the maximum probability of selecting the last success for $m = 2$ is consistent with the corresponding limit for the classical secretary problem with two selection chances.

Keywords: optimal stopping; selecting the last success; multiple selection chances

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1. Introduction

For a positive integer N, let $X_1, X_2, \ldots, X_N$ denote independent 0/1 random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We observe these X_i's sequentially and claim that the ith trial is a success if $X_i = 1$. The problem lies in finding a rule $\tau \in \mathcal{T}$ to maximize the probability of selecting the last success, where $\mathcal{T}$ is the class of all selection rules such that $\{\tau = j\} \in \sigma(X_1, X_2, \cdots, X_j)$; that is, the decision of whether to select the jth success depends on the information up to j. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ and let $p_i = \mathrm{P}(X_i = 1)$ and $q_i = 1 - p_i = \mathrm{P}(X_i = 0)$ for $i \in \mathcal{N}$, where we leave out the trivial case and assume that there exists at least one $i \in \mathcal{N}$ such that $p_i > 0$. In addition, let r_i, $i \in \mathcal{N}$, denote the odds of the ith trial; that is, $r_i = p_i/q_i$, where we set $r_i = +\infty$ if $p_i = 1$. When exactly one selection chance was allowed, Bruss [3] solved the problem with elegant simplicity as follows.

Proposition 1.1. (Theorem 1 in Bruss [3].) *Suppose that exactly one selection chance is given in the problem above. Then, the optimal selection rule* $\tau_*^{(1)}$ *selects the first success after the sum of the future odds becomes less than one; that is,*

$$\tau_*^{(1)} = \min\bigl\{i \geq i_*^{(1)} : X_i = 1\bigr\}, \tag{1.1}$$

$$i_*^{(1)} = \min\Bigl\{i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1\Bigr\}, \tag{1.2}$$

where $\min(\emptyset) = +\infty$ *and* $\sum_{j=a}^{b} \cdot = 0$ *when* $b < a$ *conventionally. Furthermore, the maximum probability of "win" (selecting the last success) is given by*

$$P^{(1)}(\mathrm{win}) = P_N^{(1)}(p_1, \ldots, p_N) = \prod_{k=i_*^{(1)}}^{N} q_k \sum_{k=i_*^{(1)}}^{N} r_k. \tag{1.3}$$

This result, referred to as the sum-the-odds theorem or odds theorem in short, is attractive because it can be applied to many basic optimal stopping problems such as betting, the classical secretary problem (CSP) and the group-interview secretary problem proposed by Hsiau and Yang [11]. Bruss [3] also proved that $P^{(1)}(\text{win})$ in (1.3) is bounded below by $R^{(1)}\, e^{-R^{(1)}}$ with $R^{(1)} = \sum_{j=i_*^{(1)}}^{N} r_j$. Remarkably, in [4], he found that it is bounded below by e^{-1} when $\sum_{j=1}^{N} r_j \geq 1$. These results generalize the known lower bounds for the CSP, where each p_i has the specific value of $p_i = 1/i$ for $i \in \mathcal{N}$ (e.g., Hill and Krengel [10]).

After Bruss [3], which includes the problem with random number of observations, the odds theorem has been extended in several directions. Bruss and Paindaveine [5] extended it to the problem of selecting the last ℓ (> 1) successes. Hsiau and Yang [12] considered the problem with Markov-dependent trials. Recently, Ferguson [8] extended the odds theorem in several ways, where infinite number of trials are allowed, the payoff for not selecting till the end is different from the payoff for selecting a success that is not the last, and the trials are generally dependent. Furthermore, he applied his extension to the stopping game of Sakaguchi [14].

In this paper, we consider yet another extension of the result by Bruss [3]; that is, we are interested in the problem with multiple selection chances. In our first main result, we derive the optimal rule for the problem of selecting the last success with m $(\in \mathcal{N})$ selection chances and express the optimal rule as a combination of multiple odds-sums. Our extension is applied to the multi-selection versions of the problems to which the odds theorem can be applied (e.g., the CSP with multiple selection chances in Gilbert and Mosteller [9] and Sakaguchi [13]). In our second main result, we provide a formula for computing the probability of win for the problem with m $(\in \mathcal{N})$ selection chances and provide the closed-form formulas for $m = 2$ and 3. Furthermore, we give the lower and upper bounds for the maximum probability of win for $m = 2$ and derive its limit as $N \to \infty$ under some condition on p_i, $i \in \mathcal{N}$. This limit of the maximum probability of win is consistent with the known limit $e^{-1} + e^{-3/2}$ for the CSP with two selection chances (e.g., Gilbert and Mosteller [9], Bruss [2], and Ano and Ando [1]).

This paper is organized as follows. In Section 2, we consider the optimal rule for the problem of selecting the last success with m ($\in \mathcal{N}$) selection chances. Our approach is essentially based on the technique of Ano and Ando [1], in which they studied the condition for the monotone (equivalent, one-step look-ahead) selection rule to be optimal in multiple selection problems. For more details on the monotone selection problem, refer to Chow et al. [6] or Ferguson [7]. In Section 3, we derive some formulas for the maximum probability of win. We give the bounds for the maximum probability of win for $m = 2$ and derive its limit as $N \to \infty$ under some condition on p_i, $i \in \mathcal{N}$. Finally, we conclude the paper by making conjectures on the limits of the maximum probability of win for $m \geq 3$ and on the lower bound for $m \geq 2$.

2. Multiple sum-the-odds theorem

Suppose that we are given m ($\in \mathcal{N}$) selection chances in the problem described in the preceding section. Let $V_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and select this success when we have at most m selection chances left. Let $W_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and ignore this success when we have at most m selection chances left. Furthermore, let $M_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and decide whether to select when we have at most m selection chances left. The optimality equation for each $m \in \mathcal{N}$ is then given by

$$M_i^{(m)} = \max\{V_i^{(m)}, W_i^{(m)}\}, \quad i \in \mathcal{N}. \tag{2.1}$$

Clearly, if $m > N - i$ (the remaining selection chances are more than the remaining observations) and we observe $X_i = 1$, then the decision to select results in win with probability 1, so that $M_i^{(m)} = V_i^{(m)} = 1$ for $i > N - m$. In particular, we have $M_N^{(m)} = V_N^{(m)} = 1$ and $W_N^{(m)} = 0$ for any $m \in \mathcal{N}$.

We observe that $V_i^{(m)}$ is represented as the sum of two conditional probabilities; the first is that no success appears in $i+1,\dots,N$ provided that $X_i=1$ and the second is that we finally win in starting at $i+1$ with $m-1$ selection chances provided that $X_i=1$. Since the latter conditional probability is equal to $W_i^{(m-1)}$, we have for each $m\in\mathcal{N}$,

$$V_i^{(m)} = \mathrm{P}(X_{i+1}=X_{i+2}=\cdots=X_N=0 \mid X_i=1) + W_i^{(m-1)}$$
$$= \prod_{j=i+1}^{N} q_j + W_i^{(m-1)}, \quad i\in\mathcal{N}, \tag{2.2}$$

where we set $W_i^{(0)} := 0$ for $i\in\mathcal{N}$ and $\prod_{j=a}^{b}\cdot = 1$ when $b<a$ conventionally. The second equality provided above follows from the independence of the X_i's. On the other hand, $W_i^{(m)}$ is given as the conditional probability with which we finally win when we make the optimal decision at the first success after $i$provided that $X_i=1$, so that, for each $m\in\mathcal{N}$,

$$W_i^{(m)} = \sum_{j=i+1}^{N} \mathrm{P}(X_{i+1}=\cdots=X_{j-1}=0, X_j=1 \mid X_i=1)\, M_j^{(m)}$$
$$= \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k\right) p_j\, M_j^{(m)}, \quad i\in\mathcal{N}. \tag{2.3}$$

As a preparatory step in studying the problem with multiple selection chances, we hereby provide another proof of the odds theorem (Proposition 1.1) using the notion of the monotone stopping rule in Chow et al. [6].

Another Proof of Proposition 1.1. We prove only the first part of Proposition 1.1. The monotone selection region for the single selection problem is given by $B^{(1)} := \{i\in\mathcal{N} : G_i^{(1)} > 0\}$, where

$$G_i^{(1)} := V_i^{(1)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k\right) p_j\, V_j^{(1)}, \quad i\in\mathcal{N}. \tag{2.4}$$

Note that $B^{(1)}$ is the region of $i\in\mathcal{N}$ such that the probability of win by selecting $X_i=1$ is greater than that by ignoring $X_i=1$ and then selecting the first success after X_i. ¿From (2.2), we have $V_i^{(1)} = \prod_{j=i+1}^{N} q_j$ and, if there exists $j \in \{i+1,\ldots,N\}$ such that $q_j = 0$, then (2.4) leads to $G_i^{(1)} \le 0$. On the other hand, if $q_j > 0$ for all $j = i+1,\ldots,N$, then (2.4) is written as

$$G_i^{(1)} = \prod_{j=i+1}^{N} q_j - \sum_{j=i+1}^{N} \Bigl(\prod_{k=i+1}^{j-1} q_k\Bigr) p_j \Bigl(\prod_{k=j+1}^{N} q_k\Bigr) = \prod_{j=i+1}^{N} q_j \Bigl(1 - \sum_{j=i+1}^{N} r_j\Bigr). \tag{2.5}$$

Therefore, if $G_i^{(1)} > 0$ for some $i \in \mathcal{N}$, then $q_j > 0$ for all $j = i+1,\ldots,N$ and (2.5) gives $\sum_{j=i+1}^{N} r_j < 1$. Conversely, if $\sum_{j=i+1}^{N} r_j < 1$ for some $i \in \mathcal{N}$, then $q_j > 0$ for all $j = i+1,\ldots,N$ and (2.5) gives $G_i^{(1)} > 0$. Namely, $G_i^{(1)} > 0$ is equivalent to $\sum_{j=i+1}^{N} r_j < 1$ and $B^{(1)}$ is given by

$$B^{(1)} = \Bigl\{ i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1 \Bigr\}.$$

Since $\sum_{j=i+1}^{N} r_j$ is clearly nonincreasing in i, $B^{(1)}$ is "closed" in the sense of the monotone problem in Chow et al [6]; that is, $i \in B^{(1)}$ implies that $j \in B^{(1)}$ for all $j = i, i+1,\ldots,N$. Hence, the optimal rule for the single selection problem is given by (1.1) and (1.2).

We can now state the optimal rules for the multiple selection problem.

Theorem 2.1. *Suppose that we have at most m $(\in \mathcal{N})$ selection chances. Then, the optimal selection rule $\tau_*^{(m)}$ is given by*

$$\tau_*^{(m)} = \min\{i \ge i_*^{(m)} : X_i = 1\}, \tag{2.6}$$

$$i_*^{(m)} = \min\{i \in \mathcal{N} : H_i^{(m)} > 0\}, \tag{2.7}$$

where $\min(\emptyset) = +\infty$ *and for each* $i \in \mathcal{N}$, $H_i^{(m)}$, $m \in \mathcal{N}$, *are recursively defined by*

$$H_i^{(1)} := 1 - \sum_{j=i+1}^{N} r_j, \tag{2.8}$$

$$H_i^{(m)} := H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m-1)}}^{N} r_j\, H_j^{(m-1)}, \quad m = 2,3,\ldots,N, \tag{2.9}$$

with $a \vee b = \max\{a, b\}$ *for* a, $b \in \mathbb{R}$. *In (2.9), if there exists a* $j \in \{i+1, \dots, N\}$ *such that* $p_j = 1$ *(that is,* $r_j = +\infty$*), then we set* $H_i^{(m)} := -\infty$. *Furthermore, we have*

$$1 \leq i_*^{(N)} \leq i_*^{(N-1)} \leq \cdots \leq i_*^{(1)} \leq N. \tag{2.10}$$

Proof. The monotone selection region for the problem with m ($\in \mathcal{N}$) selection chances is defined by $B^{(m)} := \{i \in \mathcal{N} : G_i^{(m)} > 0\}$, where

$$G_i^{(m)} := V_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.11}$$

To derive (2.6) and (2.7), it suffices to show that $B^{(m)}$ is closed and satisfies $B^{(m)} = \{i \in \mathcal{N} : H_i^{(m)} > 0\}$, which is also deduced by verifying that $G_i^{(m)} > 0$ is equivalent to $H_i^{(m)} > 0$ for each $i \in \mathcal{N}$ and that $i \mapsto H_i^{(m)}$ changes sign from nonpositive to positive at most once. On the other hand, to obtain (2.10), it suffices to show that $H_i^{(m)} \geq H_i^{(m-1)}$ for $i \in \mathcal{N}$ such that $H_i^{(m-1)} > -\infty$. We verify them by the induction on m.

While proving Proposition 1.1, we have observed that $G_i^{(1)} > 0$ is equivalent to $H_i^{(1)} > 0$ for $i \in \mathcal{N}$. In particular, if $q_j = 0$ for some $j \in \{i+1, \dots, N\}$, then $G_i^{(1)} \leq 0$, while if $q_j > 0$ for all $j = i+1, \dots, N$, then it holds that $G_i^{(1)} = (\prod_{j=i+1}^{N} q_j) \, H_i^{(1)}$ (refer to (2.5) and (2.8)). We have also observed that $i \mapsto H_i^{(1)}$ changes sign from nonpositive to positive at most once. The inequality $H_i^{(2)} \geq H_i^{(1)}$ for $i \in \mathcal{N}$ such that $H_i^{(1)} > -\infty$ is immediately obtained from (2.9); that is,

$$H_i^{(2)} - H_i^{(1)} = \sum_{j=(i+1) \vee i_*^{(1)}}^{N} r_j \, H_j^{(1)} \geq 0,$$

where the last inequality follows from $H_j^{(1)} > 0$ for $j \geq i_*^{(1)}$.

As the induction hypothesis, for $m' = 1, 2, \dots, m$ with some fixed $m \in \{1, 2, \dots, N-1\}$, we now assume the following.

(i) $G_i^{(m')} > 0$ is equivalent to $H_i^{(m')} > 0$ for every $i \in \mathcal{N}$. In particular, if $q_j = 0$ for some $j \in \{i+1, \dots, N\}$, then $G_i^{(m')} \leq 0$, and if $q_j > 0$ for all $j = i+1, \dots, N$, then it holds that $G_i^{(m')} = (\prod_{j=i+1}^{N} q_j) \, H_i^{(m')}$.

(ii) $i \mapsto H_i^{(m')}$ changes sign from nonpositive to positive at most once.

(iii) $H_i^{(m'+1)} - H_i^{(m')} \geq 0$ for $i \in \mathcal{N}$ such that $H_i^{(m')} > -\infty$.

By the induction hypothesis, $H_i^{(m)} > 0$ and equivalently $G_i^{(m)} > 0$ for $i \geq i_*^{(m)}$. Thus, by (i) above, $q_j > 0$ for all $j = i_*^{(m)}+1, \ldots, N$. Let us show (i)–(iii) above for $m' = m+1$. We first examine (i). ¿From (2.11), the monotone selection region in the case with $m+1$ selection chances is given by $B^{(m+1)} = \{i \in \mathcal{N} : G_i^{(m+1)} > 0\}$, where

$$G_i^{(m+1)} = V_i^{(m+1)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m+1)}, \quad i \in \mathcal{N}. \tag{2.12}$$

Since $V_j^{(m+1)} = V_j^{(1)} + W_j^{(m)}$ from (2.2), substituting this in (2.12), we obtain

$$\begin{aligned} G_i^{(m+1)} &= V_i^{(1)} + W_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, (V_j^{(1)} + W_j^{(m)}) \\ &= G_i^{(1)} + \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, (M_j^{(m)} - W_j^{(m)}), \end{aligned} \tag{2.13}$$

where the first term on the right-hand side is obtained from (2.4) and the second term is obtained from (2.3). By the induction hypothesis, we have $M_j^{(m)} = V_j^{(m)}$ for $j \geq i_*^{(m)}$ and $M_j^{(m)} = W_j^{(m)}$ for $j < i_*^{(m)}$ in (2.1); that is,

$$M_j^{(m)} - W_j^{(m)} = \begin{cases} V_j^{(m)} - W_j^{(m)} & \text{for } j \geq i_*^{(m)}, \\ 0 & \text{for } j < i_*^{(m)}. \end{cases}$$

Furthermore, the induction hypothesis reads (2.3) as

$$W_j^{(m)} = \sum_{\ell=j+1}^{N} \left(\prod_{k=j+1}^{\ell-1} q_k \right) p_\ell \, V_\ell^{(m)} \quad \text{for } j \geq i_*^{(m)}.$$

Therefore, from (2.11), we have

$$M_j^{(m)} - W_j^{(m)} = G_j^{(m)} \quad \text{for } j \geq i_*^{(m)},$$

substituting this in (2.13), we have

$$G_i^{(m+1)} = G_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, G_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.14}$$

Here, if $j \in \{i+1, \ldots, N\}$ exists such that $q_j = 0$, then this j is less than or equal to $i_*^{(m)}$ since $q_j > 0$ for all $j = i_*^{(m)}+1, \ldots, N$. Namely, this occurs only in the case of $i < i_*^{(m)}$, where the first term on the right-hand side of (2.14) is less than or equal to zero and the second term is equal to zero; that is, $G_i^{(m+1)} \le 0$. Conversely, suppose that $q_j > 0$ for all $j = i+1, \ldots, N$ for some $i \in \mathcal{N}$. Then, by the induction hypothesis, applying $G_i^{(m')} = (\prod_{j=i+1}^{N} q_j)\, H_i^{(m')}$ for $m' = 1$ and $m' = m$ to (2.14), we obtain

$$\begin{aligned} G_i^{(m+1)} &= \left(\prod_{j=i+1}^{N} q_j \right) H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \left(\prod_{\ell=j+1}^{N} q_\ell \right) H_j^{(m)} \\ &= \prod_{j=i+1}^{N} q_j \left(H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)} \right), \end{aligned}$$

so that (2.9) leads to

$$G_i^{(m+1)} = \left(\prod_{j=i+1}^{N} q_j \right) H_i^{(m+1)}. \tag{2.15}$$

¿From the observation above, if $G_i^{(m+1)} > 0$, then $q_j > 0$ for all $j = i+1, \ldots, N$ and (2.15) leads to $H_i^{(m+1)} > 0$. Conversely, if $H_i^{(m+1)} > 0$, then (2.9) states that $H_i^{(1)} > -\infty$; that is, $q_j > 0$ for all $j = i+1, \ldots, N$. Thus, (2.15) also leads to $G_i^{(m+1)} > 0$. Hence, we have (i) for $m' = m+1$.

Next we prove (ii). By the induction hypothesis, $H_i^{(m+1)} \ge H_i^{(m)}$ for $i \in \mathcal{N}$ such that $H_i^{(m)} > -\infty$ and $H_i^{(m)} > 0$ for $i \ge i_*^{(m)}$; that is, $H_i^{(m+1)} > 0$ for $i \ge i_*^{(m)}$. For $i < i_*^{(m)}$, we have $\sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)} = \sum_{j=i_*^{(m)}}^{N} r_j \, H_j^{(m)}$, which is invariant to i. Thus, (2.9) states that $H_i^{(m+1)}$ $(= H_i^{(1)} + \text{Constant})$ is nondecreasing in i $(< i_*^{(m)})$. Hence, $i \mapsto H_i^{(m+1)}$ changes sign from nonpositive to positive at most once, and (ii) holds for $m' = m+1$.

Finally, to prove (iii) for $m' = m+1$, we use (2.9) and take the difference between $H_i^{(m+2)}$ and $H_i^{(m+1)}$; that is,

$$\begin{aligned} H_i^{(m+2)} - H_i^{(m+1)} &= \sum_{j=(i+1)\vee i_*^{(m+1)}}^{N} r_j\, H_j^{(m+1)} - \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j\, H_j^{(m)} \\ &\geq \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j\, (H_j^{(m+1)} - H_j^{(m)}) \geq 0, \end{aligned}$$

where the first inequality follows from $H_j^{(m+1)} > 0$ for $j \geq i_*^{(m+1)}$ and $i_*^{(m+1)} \leq i_*^{(m)}$ by the induction hypothesis. The second inequality also follows from the induction hypothesis. Hence, the induction is completed and so is the proof.

Let $h_i^{(m)} := 1 - H_i^{(m)}$ for i and $m \in \mathcal{N}$. ¿From (2.9), $h_i^{(m)}$ for $m \in \mathcal{N}$ are then given by

$$\begin{aligned} h_i^{(1)} &= \sum_{j=i+1}^{N} r_j, \\ h_i^{(m)} &= \sum_{j=i+1}^{i_*^{(m-1)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(m-1)}}^{N} r_j\, h_j^{(m-1)}, \quad m = 2, 3, \ldots. \end{aligned}$$

We can observe from the above equations that each $h_i^{(m)}$ is expressed as a combination of multiple odds-sums. For instance, $h_i^{(2)}$ and $h_i^{(3)}$ are calculated as

$$\begin{aligned} h_i^{(2)} &= \sum_{j=i+1}^{i_*^{(1)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k, \\ h_i^{(3)} &= \sum_{j=i+1}^{i_*^{(2)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(2)}}^{N} r_j \left\{ \sum_{k=j+1}^{i_*^{(1)}-1} r_k + \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k \sum_{\ell=k+1}^{N} r_\ell \right\}. \end{aligned} \tag{2.16}$$

The optimal rule for the problem with m $(\in \mathcal{N})$ selection chances then reduces to $\tau_*^{(m)} = \min\{i \in \mathcal{N} : h_i^{(m)} < 1 \;\&\; X_i = 1\}$. Hence, we call Theorem 2.1 "multiple sum-the-odds theorem" or "multiple odds theorem" in short.

3. Maximum probability of win

In this section, we first derive a formula for computing the maximum probability of win under the optimal rule with m ($\in \mathcal{N}$) selection chances and then provide closed-form formulas for $m = 2$ and 3. Then, we give its lower and upper bounds and the limit as $N \to \infty$ for $m = 2$.

Theorem 3.1. *For the problem with at most m ($\in \mathcal{N}$) selection chances, the maximum probability of win under the optimal rule, $P^{(m)}(\text{win}) = P_N^{(m)}(p_1, \ldots, p_N)$, is given by*

$$P^{(m)}(\text{win}) = \prod_{j=i_*^{(m)}}^{N} q_j \sum_{j=i_*^{(m)}}^{N} r_j + \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j} q_k \right) r_j \, W_j^{(m-1)}, \tag{3.1}$$

where if $p_{i_*^{(m)}} = 1$, then $P^{(m)}(\text{win}) = \prod_{k=i_*^{(m)}+1}^{N} q_k + W_{i_*^{(m)}}^{(m-1)}$ (note that $p_j < 1$ for all $j = i_*^{(m)} + 1, \ldots, N$). Especially, for $m = 2$ and 3,

$$P^{(2)}(\text{win}) = \prod_{j=i_*^{(2)}}^{N} q_j \sum_{j=i_*^{(2)}}^{N} r_j \left(1 + \prod_{k=j+1}^{i_*^{(1)}-1} (1 + r_k) \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k \right), \tag{3.2}$$

$$\begin{aligned} P^{(3)}(\text{win}) = \prod_{j=i_*^{(3)}}^{N} q_j \sum_{j=i_*^{(3)}}^{N} r_j \Bigg[1 + \prod_{k=j+1}^{i_*^{(2)}-1} (1 + r_k) \\ \times \sum_{k=(j+1)\vee i_*^{(2)}}^{N} r_k \left(1 + \prod_{\ell=k+1}^{i_*^{(1)}-1} (1 + r_\ell) \sum_{\ell=(k+1)\vee i_*^{(1)}}^{N} r_\ell \right) \Bigg]. \end{aligned} \tag{3.3}$$

Proof. Note that the independence of X_i's leads to $P^{(m)}(\text{win}) = W_{i_*^{(m)}-1}^{(m)}$ under the optimal selection rule. Thus, from (2.2) and (2.3), we obtain

$$\begin{aligned} P^{(m)}(\text{win}) &= \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j-1} q_k \right) p_j \, M_j^{(m)} \\ &= \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j-1} q_k \right) p_j \left(\prod_{\ell=j+1}^{N} q_\ell + W_j^{(m-1)} \right), \end{aligned}$$

where the second equality follows from $M_j^{(m)} = V_j^{(m)}$ for $j \geq i_*^{(m)}$. Hence, (3.1) is easily obtained.

$P^{(2)}(\text{win})$ and $P^{(3)}(\text{win})$ are derived from straightforward calculations. Since the optimal rule requires the selection of the first success after $i_*^{(1)}$, we have $M_k^{(1)} = V_k^{(1)} = \prod_{\ell=k+1}^N q_k$ for $k \geq i_*^{(1)}$. It then follows from (2.3) that

$$W_j^{(1)} = \sum_{k=j+1}^{N} \left(\prod_{\ell=j+1}^{k-1} q_\ell \right) p_k \, M_k^{(1)} = \prod_{\ell=j+1}^{N} q_\ell \sum_{k=j+1}^{N} r_k \quad \text{for } j \geq i_*^{(1)} - 1.$$

On the other hand, for $j < i_*^{(1)} - 1$, we have $W_j^{(1)} = W_{i_*^{(1)}-1}^{(1)} = \prod_{\ell=i_*^{(1)}}^N q_\ell \sum_{j=i_*^{(1)}}^N r_j$. Therefore, for each $j \in \mathcal{N}$,

$$W_j^{(1)} = \prod_{\ell=(j+1)\vee i_*^{(1)}}^{N} q_\ell \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k.$$

Substituting this in (3.1) with $m = 2$ and using $1/q_k = 1 + r_k$, we have (3.2).

Using an approach similar to the one used above, we obtain

$$W_j^{(2)} = \prod_{\ell=(j+1)\vee i_*^{(2)}}^{N} q_\ell \sum_{k=(j+1)\vee i_*^{(2)}}^{N} r_k \left(1 + \prod_{\ell=k+1}^{i_*^{(1)}-1} (1 + r_\ell) \sum_{\ell=(k+1)\vee i_*^{(1)}}^{N} r_\ell \right).$$

Substituting this in (3.1) with $m = 3$, we have (3.3).

Next, we consider the lower and upper bounds for the maximum probability of win for $m = 2$ and its limit as $N \to \infty$. In the following, to emphasize the dependence on N, we subscript "N" and write $P_N^{(m)}(\text{win})$ and $i_{*,N}^{(m)}$ occasionally. Let $R_N^{(m)} = \sum_{j=i_{*,N}^{(m)}}^N r_j$ and $R_N^{(m,2)} = \sum_{j=i_{*,N}^{(m)}}^N r_j^2$ for $m \in \mathcal{N}$. Note from (1.2) and (2.10) that $0 < \min(1, \sum_{j=1}^N r_j) \leq R_N^{(1)} \leq R_N^{(2)} \leq \cdots \leq R_N^{(N)} \leq \sum_{j=1}^N r_j$. For the single selection problem, Bruss [3] deduced that

$$R_N^{(1)} \, e^{-R_N^{(1)}} < P_N^{(1)}(\text{win}) \leq R_N^{(1)} \, e^{-R_N^{(1)} + R_N^{(1,2)}},$$

and further proved that, if $R_N^{(1)} \to 1$ and $R_N^{(1,2)} \to 0$ as $N \to \infty$, then

$$P_N^{(1)}(\text{win}) \to 1/e \quad \text{as } N \to \infty.$$

For the double selection problem, we give the bounds and the limit as $N \to \infty$ for the maximum probability of win. We observe that our limit $e^{-1} + e^{-3/2}$ is the same as that for the CSP with two selection chances under a reasonable condition on $R_N^{(m)}$ and $R_N^{(m,2)}$ as $N \to \infty$ (e.g., Gilbert and Mosteller [9], Bruss [2], and Ano and Ando [1]).

Theorem 3.2. *For the maximum probability of win with $m = 2$, we have*

$$P_N^{(2)}(\text{win}) > R_N^{(1)}\, e^{-R_N^{(1)}} + e^{-R_N^{(2)}}, \tag{3.4}$$

$$\begin{aligned} P_N^{(2)}(\text{win}) < R_N^{(1)}\, e^{-R_N^{(1)}+R_N^{(1,2)}} \\ + (1 + r_{i_*^{(1)}}\, R_N^{(1)} + r_{i_*^{(2)}})\, e^{-R_N^{(2)}+R_N^{(2,2)}}. \end{aligned} \tag{3.5}$$

Furthermore, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$, $R_N^{(1,2)} \to 0$ and $R_N^{(2,2)} \to 0$ as $N \to \infty$, then

$$P_N^{(2)}(\text{win}) \to e^{-1} + e^{-3/2} \quad \text{as } N \to \infty. \tag{3.6}$$

Proof. We first derive the lower bound of (3.4). A simple expansion of (3.2) in Theorem 3.1 yields

$$\begin{aligned} P^{(2)}(\text{win}) = R^{(2)} \prod_{j=i_*^{(2)}}^{N} q_j + R^{(1)} \sum_{j=i_*^{(2)}}^{i_*^{(1)}-1} \left(\prod_{k=i_*^{(2)}}^{j-1} q_k \right) p_j \left(\prod_{k=i_*^{(1)}}^{N} q_k \right) \\ + \prod_{j=i_*^{(2)}}^{N} q_j \sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k, \end{aligned} \tag{3.7}$$

where the subscript "N" is omitted to simplify the notation. In the second term on the right-hand side (RHS) above, we note that $\sum_{j=i_*^{(2)}}^{i_*^{(1)}-1} \big(\prod_{k=i_*^{(2)}}^{j-1} q_k\big)\, p_j = 1 - \prod_{j=i_*^{(2)}}^{i_*^{(1)}-1} q_j$ since it represents the probability that at least one success appears from $i_*^{(2)}$ to $i_*^{(1)} - 1$ when $i_*^{(1)} > i_*^{(2)}$ (while it is equal to zero when $i_*^{(1)} = i_*^{(2)}$). Thus, we obtain

$$\begin{aligned} (\text{2nd term on RHS of (3.7)}) &= R^{(1)} \left(1 - \prod_{j=i_*^{(2)}}^{i_*^{(1)}-1} q_j \right) \prod_{k=i_*^{(1)}}^{N} q_k \\ &= R^{(1)} \left(\prod_{j=i_*^{(1)}}^{N} q_j - \prod_{j=i_*^{(2)}}^{N} q_j \right). \end{aligned} \tag{3.8}$$

Consider the third term on the right-hand side in (3.7). Since $h_i^{(2)} = 1 - H_i^{(2)} \geq 1$ for $i < i_*^{(2)}$, substituting $i = i_*^{(2)} - 1$ in (2.16), we have $\sum_{j=i_*^{(2)}}^{i_*^{(1)}-1} r_j + \sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k \geq 1$, which is equivalent to

$$\sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k \geq 1 + R^{(1)} - R^{(2)}.$$

Therefore, we obtain

$$(\text{3rd term on RHS of (3.7)}) \geq (1 + R^{(1)} - R^{(2)}) \prod_{j=i_*^{(2)}}^{N} q_j. \tag{3.9}$$

Substituting (3.8) and (3.9) in (3.7) yields

$$P^{(2)}(\text{win}) \geq R^{(1)} \prod_{j=i_*^{(1)}}^{N} q_j + \prod_{j=i_*^{(2)}}^{N} q_j. \tag{3.10}$$

Here, noting that $1/q_j = 1 + r_j$ and taking the logarithm, we have for any $s \in \mathcal{N}$,

$$\log \prod_{j=s}^{N} q_j = -\sum_{j=s}^{N} \log(1 + r_j) > -\sum_{j=s}^{N} r_j,$$

where the inequality follows since $\log(1+x) \leq x$ for $x \geq 0$; the equality follows only when $x = 0$. Hence, we obtain $\prod_{j=s}^{N} q_j > e^{-R}$ with $R = \sum_{j=s}^{N} r_j$. Substituting this in (3.10) with $s = i_*^{(1)}$ and $s = i_*^{(2)}$, we obtain (3.4).

Next we derive the upper bound of (3.5). For this, we examine the third term on the right-hand side in (3.7). Since $h_i^{(2)} < 1$ for $i \geq i_*^{(2)}$, substituting $i = i_*^{(2)}$ in (2.16), we obtain $\sum_{j=i_*^{(2)}+1}^{i_*^{(1)}-1} r_j + \sum_{j=(i_*^{(2)}+1)\vee i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k < 1$, so that,

$$\sum_{j=i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k < 1 + (1 + r_{i_*^{(1)}})\, R^{(1)} - (R^{(2)} - r_{i_*^{(2)}}).$$

Therefore, we obtain

$$(\text{3rd term on RHS of (3.7)}) < \big(1 + (1 + r_{i_*^{(1)}})\, R^{(1)} - R^{(2)} + r_{i_*^{(2)}}\big) \prod_{j=i_*^{(2)}}^{N} q_j. \tag{3.11}$$

Applying (3.8) and (3.11) to (3.7), we obtain

$$P^{(2)}(\text{win}) < R^{(1)} \prod_{j=i_*^{(1)}}^{N} q_j + (1 + r_{i_*^{(1)}} R^{(1)} + r_{i_*^{(2)}}) \prod_{j=i_*^{(1)}}^{N} q_j. \tag{3.12}$$

Here, since $1/q_j = 1 + r_j$, using $\log(1+x) \geq x - x^2$ for $x \geq 0$, we obtain for any $s \in \mathcal{N}$,

$$\log \prod_{j=s}^{N} q_j \leq -\sum_{j=s}^{N} r_j + \sum_{j=s}^{N} r_j^2.$$

Hence, by assigning $\sum_{j=s}^{N} r_j = R$ and $\sum_{j=s}^{N} r_j^2 = R'$, we obtain $\prod_{j=s}^{N} q_j \leq e^{-R+R'}$. Applying this in (3.12) with $s = i_*^{(1)}$ and $s = i_*^{(2)}$, we obtain (3.5).

Finally, we have $r_{i_{*,N}^{(1)}} \to 0$ and $r_{i_{*,N}^{(2)}} \to 0$ as $N \to \infty$, since $R_N^{(1,2)} \to 0$ and $R_N^{(2,2)} \to 0$ as $N \to \infty$, respectively. Therefore, (3.4) and (3.5) yield (3.6) as $N \to \infty$.

As a final remark, in the multiple selection problem, we make two conjectures on the limits and lower bounds for the maximum probability of win. First, we conjecture that, if $R_N^{(m)}$ and $R_N^{(m,2)}$, $m = 1, 2, \ldots$, have the same limits as those for the CSP with multiple selection chances, then the limit of the maximum probability of win is also consistent with that for the CSP; that is,

$$\lim_{N\to\infty} P_N^{(m)}(\text{win}) = \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N} \quad \text{for } m = 1, 2, \ldots.$$

The case of $m = 1$ was solved by Bruss [3] and the case of $m = 2$ is solved above. For instance, for the triple selection problem, our conjecture states that, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$ and $R_N^{(3)} \to 47/24$ with $R_N^{(m,2)} \to 0$, $m = 1, 2, 3$ as $N \to \infty$, then

$$\lim_{N\to\infty} P_N^{(3)}(\text{win}) = e^{-1} + e^{-3/2} + e^{-47/24}.$$

On performing some delicate and complicated calculations, this triple selection case could be confirmed by an approach similar to that for $P_N^{(2)}(\text{win})$. However, the problem of general m is more challenging.

Second, for the lower bounds for the maximum probability of win, our conjecture is stated as that, for some reasonable condition on p_i, $i \in \mathcal{N}$,

$$P_N^{(m)}(\text{win}) > \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N} \quad \text{for } m = 1, 2, \ldots.$$

For this problem, the case of $m = 1$ was solved by Bruss [4]. However, the case of $m = 2$ is still open.

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