Odds theorem with multiple selection chances

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What is our stopping problem?—1/13

1. For a positive integer N, let $X_1, X_2, \ldots, X_N$ denote independent 0/1 random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.
2. We observe these X_i's sequentially and claim that the ith trial is a success if $X_i = 1$.
3. The problem lies in finding a rule $\tau \in \mathcal{T}$ to maximize the probability of selecting the last success, where $\mathcal{T}$ is the class of all selection rules such that $\{\tau = j\} \in \sigma(X_1, X_2, \cdots, X_j)$.
4. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ and let $p_i = \mathrm{P}(X_i = 1)$ and $q_i = 1 - p_i$ for $i \in \mathcal{N}$, where we leave out the trivial case and assume that there exists at least one $i \in \mathcal{N}$ such that $p_i > 0$. In addition, let r_i, $i \in \mathcal{N}$, denote the odds of the ith trial; that is, $r_i = p_i/q_i$, where we set $r_i = +\infty$ if $p_i = 1$.

Odds theorem with single selection chance—2/13

1. When exactly one selection chance was allowed, Bruss [1] solved the problem with elegant simplicity as follows.
2. The optimal selection rule $\tau_*^{(1)}$ selects the first success after the sum of the future odds becomes less than one; that is,

$$\tau_*^{(1)} = \min\{i \geq i_*^{(1)} : X_i = 1\}, \tag{1.1}$$

$$i_*^{(1)} = \min\Big\{i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1\Big\}, \tag{1.2}$$

where $\min(\emptyset) = +\infty$ and $\sum_{j=a}^{b} \cdot = 0$ when $b < a$ conventionally.
3. Furthermore, the maximum probability of "win" (selecting the last success) is given by

$$P^{(1)}(\text{win}) = P_N^{(1)}(p_1, \ldots, p_N) = \prod_{k=i_*^{(1)}}^{N} q_k \sum_{k=i_*^{(1)}}^{N} r_k. \tag{1.3}$$

Related works—$3/13$

1. Applicable to many basic optimal stopping problems such as betting, the classical secretary problem (CSP) and the group-interview secretary problem proposed by Hsiau and Yang (2000).
2. Bruss (2003) also found that $P^{(1)}(\mathrm{win})$ is bounded below by e^{-1} when $\sum_{j=1}^{N} r_j \geq 1$. These results generalize the known lower bounds for the CSP, where each p_i has the specific value of $p_i = 1/i$ for $i \in \mathcal{N}$ (e.g., Hill and Krengel (1992)).
3. Bruss and Paindaveine (2000) extended it to the problem of selecting the last ℓ (> 1) successes. Hsiau and Yang (2003) considered the problem with Markov-dependent trials.
4. Ferguson (2008) extended the odds theorem in several ways, where infinite number of trials are allowed and the trials are generally dependent.
5. Ano, Kakie and Miyoshi (2010) study the multiple selection problem in Markov-dependent trials.

Main results— 4/13

1. First main result: we derive the optimal rule for the problem of selecting the last success with at most m ($\in \mathcal{N}$) selection chances and express the optimal rule as a combination of multiple odds-sums.
2. Second main result: we provide a formula for computing the probability of win for the problem with m ($\in \mathcal{N}$) selection chances and provide the closed-form formulas for $m = 2$ and 3.
3. Furthermore, we give the lower and upper bounds for the maximum probability of win for $m = 2$ and derive its limit as $N \to \infty$ under some condition on p_i, $i \in \mathcal{N}$. This limit of the maximum probability of win is consistent with the known limit $e^{-1} + e^{-3/2}$ for the CSP with two selection chances.

Formulation— 5/13

1. Let $V_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and select this success when we have at most m selection chances left.
2. Let $W_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and ignore this success when we have at most m selection chances left.
3. Furthermore, let $M_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and decide whether to select when we have at most m selection chances left. The optimality equation for each $m \in \mathcal{N}$ is then given by
$$M_i^{(m)} = \max\{V_i^{(m)}, W_i^{(m)}\}, \quad i \in \mathcal{N}. \tag{2.1}$$
Clearly, if $m > N - i$ (the remaining selection chances are more than the remaining observations) and we observe $X_i = 1$, then the decision to select results in win with probability 1, so that $M_i^{(m)} = V_i^{(m)} = 1$ for $i > N - m$. In particular, we have $M_N^{(m)} = V_N^{(m)} = 1$ and $W_N^{(m)} = 0$ for any $m \in \mathcal{N}$.

Formulation (Continued)— 6/13

1. We have for each $m \in \mathcal{N}$,

$$V_i^{(m)} = \mathrm{P}(X_{i+1} = X_{i+2} = \cdots = X_N = 0 \mid X_i = 1) + W_i^{(m-1)}$$
$$= \prod_{j=i+1}^{N} q_j + W_i^{(m-1)}, \quad i \in \mathcal{N}, \tag{2.2}$$

where we set $W_i^{(0)} := 0$ for $i \in \mathcal{N}$ and $\prod_{j=a}^{b} \cdot = 1$ when $b < a$ conventionally.

2. For each $m \in \mathcal{N}$,

$$W_i^{(m)} = \sum_{j=i+1}^{N} \mathrm{P}(X_{i+1} = \cdots = X_{j-1} = 0, X_j = 1 \mid X_i = 1)\, M_j^{(m)}$$
$$= \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j\, M_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.3}$$

Another proof of the odds theorem with single selection— 7/13

1. The monotone selection region for the single selection problem is given by $B^{(1)} := \{i \in \mathcal{N} : G_i^{(1)} > 0\}$, where

$$G_i^{(1)} := V_i^{(1)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(1)} = \prod_{j=i+1}^{N} q_j \left(1 - \sum_{j=i+1}^{N} r_j \right), \quad i \in \mathcal{N}. \tag{2.4}$$

2. $G_i^{(1)} > 0$ is equivalent to $\sum_{j=i+1}^{N} r_j < 1$ and $B^{(1)}$ is given by

$$B^{(1)} = \left\{ i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1 \right\}.$$

Since $\sum_{j=i+1}^{N} r_j$ is clearly nonincreasing in i, $B^{(1)}$ is “closed” in the sense of the monotone problem in Chow et al (1971) of the classical optimal stopping theory. Hence, the optimal rule for the single selection problem is given by (1.1) and (1.2).

Multiple Sum-the-Odds Theorem— 8/13

1. The optimal selection rule $\tau_*^{(m)}$ is given by
$$\tau_*^{(m)} = \min\{i \geq i_*^{(m)} : X_i = 1\}, \tag{2.5}$$
$$i_*^{(m)} = \min\{i \in \mathcal{N} : H_i^{(m)} > 0\}, \tag{2.6}$$
where $\min(\emptyset) = +\infty$ and for each $i \in \mathcal{N}$, $H_i^{(m)}$, $m \in \mathcal{N}$, are recursively defined by $H_i^{(1)} := 1 - \sum_{j=i+1}^{N} r_j$ and
$$H_i^{(m)} := H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m-1)}}^{N} r_j\, H_j^{(m-1)}, \quad m = 2, 3, \ldots, N, \tag{2.7}$$
with $a \vee b = \max\{a, b\}$ for a, $b \in \mathbb{R}$. In (2.7), if there exists a $j \in \{i+1, \ldots, N\}$ such that $p_j = 1$ (that is, $r_j = +\infty$), then we set $H_i^{(m)} := -\infty$.
2. Furthermore, we have
$$1 \leq i_*^{(m)} \leq i_*^{(m-1)} \leq \cdots \leq i_*^{(1)} \leq N. \tag{2.8}$$

Proof of Multiple Sum-the-Odds Theorem— 9/13

1. The monotone selection region for the problem with m $(\in \mathcal{N})$ selection chances is defined by $B^{(m)} := \{i \in \mathcal{N} : G_i^{(m)} > 0\}$, where

$$G_i^{(m)} := V_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.9}$$

2. Key of the proof is the following equation on m;

$$G_i^{(m+1)} = \left(\prod_{j=i+1}^{N} q_j \right) H_i^{(m+1)}.$$

$$H_i^{(m+1)} = H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)}. \tag{2.10}$$

3. We verify the optimality of $B^{(m)}$ by the induction on m.

Probability of win— 10/13

1. For the problem with at most m $(\in \mathcal{N})$ selection chances, the maximum probability of win under the optimal rule, $P^{(m)}(\mathrm{win}) = P_N^{(m)}(p_1, \ldots, p_N)$, is given by

$$P^{(m)}(\mathrm{win}) = \prod_{j=i_*^{(m)}}^{N} q_j \sum_{j=i_*^{(m)}}^{N} r_j + \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j} q_k \right) r_j \, W_j^{(m-1)}, \qquad (3.1)$$

where if $p_{i_*^{(m)}} = 1$, then $P^{(m)}(\mathrm{win}) = \prod_{k=i_*^{(m)}+1}^{N} q_k + W_{i_*^{(m)}}^{(m-1)}$ (note that $p_j < 1$ for all $j = i_*^{(m)} + 1, \ldots, N$).

2. Especially, for $m = 2$ and 3,

$$P^{(2)}(\mathrm{win}) = \prod_{j=i_*^{(2)}}^{N} q_j \sum_{j=i_*^{(2)}}^{N} r_j \left(1 + \prod_{k=j+1}^{i_*^{(1)}-1} (1 + r_k) \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k \right), \qquad (3.2)$$

$$P^{(3)}(\mathrm{win}) = \prod_{j=i_*^{(3)}}^{N} q_j \sum_{j=i_*^{(3)}}^{N} r_j \left[1 + \prod_{k=j+1}^{i_*^{(2)}-1} (1 + r_k) \times \cdots \right. . \qquad (3.3)$$

Limiting probability of win— 11/13

1. In the following, to emphasize the dependence on N, we subscript "N" and write $P_N^{(m)}(\text{win})$ and $i_{*,N}^{(m)}$ occasionally. Let $R_N^{(m)} = \sum_{j=i_{*,N}^{(m)}}^N r_j$ and $R_N^{(m,2)} = \sum_{j=i_{*,N}^{(m)}}^N r_j^2$ for $m \in \mathcal{N}$.
2. For the maximum probability of win with $m = 2$, we have

$$P_N^{(2)}(\text{win}) > R_N^{(1)}\, e^{-R_N^{(1)}} + e^{-R_N^{(2)}}, \tag{3.4}$$

$$\begin{aligned} P_N^{(2)}(\text{win}) &< R_N^{(1)}\, e^{-R_N^{(1)} + R_N^{(1,2)}} \\ &\quad + (1 + r_{i_*^{(1)}}\, R_N^{(1)} + r_{i_*^{(2)}})\, e^{-R_N^{(2)} + R_N^{(2,2)}}. \end{aligned} \tag{3.5}$$

Furthermore, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$, $R_N^{(1,2)} \to 0$ and $R_N^{(2,2)} \to 0$ as $N \to \infty$, then

$$P_N^{(2)}(\text{win}) \to e^{-1} + e^{-3/2} \quad \text{as } N \to \infty. \tag{3.6}$$

Conjectures on the probability of win — 12/13

1. First, we conjecture that, if $R_N^{(m)}$ and $R_N^{(m,2)}$, $m = 1, 2, \ldots$, have the same limits as those for the CSP with multiple selection chances, then the limit of the maximum probability of win is also consistent with that for the CSP; that is,

$$\lim_{N\to\infty} P_N^{(m)}(\mathrm{win}) = \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N} \quad \text{for } m = 1, 2, \ldots.$$

For instance, for the triple selection problem, our conjecture states that, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$ and $R_N^{(3)} \to 47/24$ with $R_N^{(m,2)} \to 0$, $m = 1, 2, 3$ as $N \to \infty$, then

$$\lim_{N\to\infty} P_N^{(3)}(\mathrm{win}) = e^{-1} + e^{-3/2} + e^{-47/24}.$$

2. Second, for the lower bounds for the maximum probability of win, our conjecture is stated as that, for some reasonable condition on p_i, $i \in \mathcal{N}$,

$$P_N^{(m)}(\mathrm{win}) > \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N} \quad \text{for } m = 1, 2, \ldots.$$

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