Multiple stopping version of the odds theorem in optimal stopping

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What is our stopping problem? —1/20

1. For a positive integer N, let $X_1, X_2, \ldots, X_N$ denote independent 0/1 random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.
2. We observe these X_i's sequentially and claim that the ith trial is a success if $X_i = 1$.
3. The problem lies in finding a rule $\tau \in \mathcal{T}$ to maximize the probability of selecting the last success, where $\mathcal{T}$ is the class of all selection rules such that $\{\tau = j\} \in \sigma(X_1, X_2, \cdots, X_j)$.
4. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ and let $p_i = \mathrm{P}(X_i = 1)$ and $q_i = 1 - p_i$ for $i \in \mathcal{N}$, where we leave out the trivial case and assume that there exists at least one $i \in \mathcal{N}$ such that $p_i > 0$. In addition, let r_i, $i \in \mathcal{N}$, denote the odds of the ith trial; that is, $r_i = p_i/q_i$, where we set $r_i = +\infty$ if $p_i = 1$.

Odds theorem with single selection chance —2/20

1. When exactly one selection chance was allowed, Bruss [1] solved the problem with elegant simplicity as follows.
2. **The optimal selection rule** $\tau_*^{(1)}$ selects the first success after the sum of the future odds becomes less than one; that is,

$$\tau_*^{(1)} = \min\{i \geq i_*^{(1)} : X_i = 1\}, \tag{1.1}$$

$$i_*^{(1)} = \min\Big\{i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1\Big\}, \tag{1.2}$$

   where $\min(\emptyset) = +\infty$ and $\sum_{j=a}^{b} \cdot = 0$ when $b < a$ conventionally.
3. Furthermore, **the maximum probability of "win"** (selecting the last success) is given by

$$P^{(1)}(\mathrm{win}) = P_N^{(1)}(p_1, \ldots, p_N) = \prod_{k=i_*^{(1)}}^{N} q_k \sum_{k=i_*^{(1)}}^{N} r_k. \tag{1.3}$$

Related works —3/20

1. Applicable to many basic optimal stopping problems such as betting, the Classical Secretary Problem (CSP) and the group-interview secretary problem proposed by Hsiau and Yang (2000).
2. Bruss (2003) remarkabley found that $P^{(1)}(\mathrm{win})$ is bounded below by e^{-1} for any $\{p_i\}_{i=1}^N$ when $\sum_{j=1}^N r_j \geq 1$. These results generalize the known lower bounds for the CSP, where each p_i has the specific value of $p_i = 1/i$ for $i \in \mathcal{N}$ (e.g., Hill and Krengel (1992)).
3. Bruss and Paindaveine (2000) extended it to the problem of selecting the last ℓ (> 1) successes. Hsiau and Yang (2003) considered the problem with Markov-dependent trials. Ferguson (2008) extended the odds theorem in several ways, where infinite number of trials are allowed and the trials are generally dependent.
4. Ano, Kakie and Miyoshi (2010) study the multiple selection problem in Markov-dependent trials. Historically, the multiple selection CSP goes back to Gilbert and Mosteller (1966).

Main results — 4/20

1. **First main result:** we derive the optimal rule for the problem of selecting the last success with at most m $(\in \mathcal{N})$ selection chances and express the optimal rule as a combination of multiple odds-sums.
2. **Second main result:** we provide a formula for computing the probability of win for the problem with m $(\in \mathcal{N})$ selection chances and provide the closed-form formulas for $m = 2$ and 3.
3. **Furthermore,** we give the lower and upper bounds for the maximum probability of win for $m = 2$ and derive its limit as $N \to \infty$ under some condition on p_i, $i \in \mathcal{N}$. This limit of the maximum probability of win is consistent with the known limit $e^{-1} + e^{-3/2}$ for the CSP with two selection chances.

Formulation — 5/20

1. $V_i^{(m)}$, $i \in \mathcal{N}$:= the conditional maximum probability of win provided that we observe $X_i = 1$ and select this success when we have at most m selection chances left.
2. $W_i^{(m)}$, $i \in \mathcal{N}$:=the conditional maximum probability of win provided that we observe $X_i = 1$ and ignore this success when we have at most m selection chances left.
3. Furthermore, let $M_i^{(m)}$, $i \in \mathcal{N}$, denote the conditional maximum probability of win provided that we observe $X_i = 1$ and decide whether to select when we have at most m selection chances left. The optimality equation for each $m \in \mathcal{N}$ is then given by

$$M_i^{(m)} = \max\{V_i^{(m)}, W_i^{(m)}\}, \quad i \in \mathcal{N}. \tag{2.1}$$

4. Boundary conditions:Clearly, if $m > N - i$ (the remaining selection chances are more than the remaining observations) and we observe $X_i = 1$, then the decision to select results in win with probability 1, so that $M_i^{(m)} = V_i^{(m)} = 1$ for $i > N - m$. In particular, we have $M_N^{(m)} = V_N^{(m)} = 1$ and $W_N^{(m)} = 0$ for any $m \in \mathcal{N}$.

Formulation (Continued) — 6/20

1. We have for each $m \in \mathcal{N}$,

$$V_i^{(m)} = \mathrm{P}(X_{i+1} = X_{i+2} = \cdots = X_N = 0 \mid X_i = 1) + W_i^{(m-1)}$$
$$= \prod_{j=i+1}^{N} q_j + W_i^{(m-1)}, \quad i \in \mathcal{N}, \tag{2.2}$$

where we set $W_i^{(0)} := 0$ for $i \in \mathcal{N}$ and $\prod_{j=a}^{b} \cdot = 1$ when $b < a$ conventionally.

2. For each $m \in \mathcal{N}$,

$$W_i^{(m)} = \sum_{j=i+1}^{N} \mathrm{P}(X_{i+1} = \cdots = X_{j-1} = 0, X_j = 1 \mid X_i = 1)\, M_j^{(m)}$$
$$= \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j\, M_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.3}$$

Another proof of the odds theorem with single selection — 7/20

1. The monotone selection region for the single selection problem is given by $B^{(1)} := \{i \in \mathcal{N} : G_i^{(1)} > 0\}$, where

$$G_i^{(1)} := V_i^{(1)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(1)} = \prod_{j=i+1}^{N} q_j \left(1 - \sum_{j=i+1}^{N} r_j \right), \quad i \in \mathcal{N}. \tag{2.4}$$

2. $G_i^{(1)} > 0$ is equivalent to $\sum_{j=i+1}^{N} r_j < 1$ and $B^{(1)}$ is written as

$$B^{(1)} = \left\{ i \in \mathcal{N} : \sum_{j=i+1}^{N} r_j < 1 \right\}.$$

Since $\sum_{j=i+1}^{N} r_j$ is clearly nonincreasing in i, $B^{(1)}$ is "closed" (that is, $\mathrm{P}(X_j \in B^{(1)} | X_i \in B^{(1)}) = 1$ for any $j > i$) in the sense of the monotone problem in Chow et al (1971) of the optimal stopping theory. Hence, the optimal rule for the single selection problem is given by (1.1) and (1.2).

Multiple Sum-the-Odds Theorem — 8/20

1. The optimal selection rule $\tau_*^{(m)}$ is given by

$$\tau_*^{(m)} = \min\{i \geq i_*^{(m)} : X_i = 1\}, \tag{2.5}$$

$$i_*^{(m)} = \min\{i \in \mathcal{N} : H_i^{(m)} > 0\}, \tag{2.6}$$

where $\min(\emptyset) = +\infty$ and for each $i \in \mathcal{N}$, $H_i^{(m)}$, $m \in \mathcal{N}$, are recursively defined by $H_i^{(1)} := 1 - \sum_{j=i+1}^{N} r_j$ and

$$H_i^{(m)} := H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m-1)}}^{N} r_j \, H_j^{(m-1)}, \quad m = 2, 3, \ldots, N, \tag{2.7}$$

with $a \vee b = \max\{a, b\}$ for a, $b \in \mathbb{R}$. In (2.7), if there exists a $j \in \{i+1, \ldots, N\}$ such that $p_j = 1$ (that is, $r_j = +\infty$), then we set $H_i^{(m)} := -\infty$. Note that $H_i^{(m)}$ is expressed by the multiple sums the odds.

2. Furthermore, we have

$$1 \leq i_*^{(m)} \leq i_*^{(m-1)} \leq \cdots \leq i_*^{(1)} \leq N. \tag{2.8}$$

Proof of Multiple Sum-the-Odds Theorem 1/7 — 9/20

1. The monotone selection region for the problem with m $(\in \mathcal{N})$ selection chances is defined by $B^{(m)} := \{i \in \mathcal{N} : G_i^{(m)} > 0\}$, where

$$G_i^{(m)} := V_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m)}, \quad i \in \mathcal{N}. \qquad (2.9)$$

2. Key of the proof is the following recursive equation on m;

$$G_i^{(m+1)} = \left(\prod_{j=i+1}^{N} q_j \right) H_i^{(m+1)}.$$

$$H_i^{(m+1)} = H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)}. \qquad (2.10)$$

3. We can verify the optimality of $B^{(m)}$ by the induction on m.

Proof of Multiple Sum-the-Odds Theorem 2/7 — 10/20

1. The monotone selection region for the problem with m ($\in \mathcal{N}$) selection chances is defined by $B^{(m)} := \{i \in \mathcal{N} : G_i^{(m)} > 0\}$, where
$$G_i^{(m)} := V_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.11}$$
To derive (2.5) and (2.6), it suffices to show that $B^{(m)}$ is closed and satisfies $B^{(m)} = \{i \in \mathcal{N} : H_i^{(m)} > 0\}$, which is also deduced by verifying that $G_i^{(m)} > 0$ is equivalent to $H_i^{(m)} > 0$ for each $i \in \mathcal{N}$ and that $i \mapsto H_i^{(m)}$ changes sign from nonpositive to positive at most once.
2. On the other hand, to obtain (2.8), it suffices to show that $H_i^{(m)} \geq H_i^{(m-1)}$ for $i \in \mathcal{N}$ such that $H_i^{(m-1)} > -\infty$. We verify them by the induction on m.

Proof of Multiple Sum-the-Odds Theorem 3/7 — 11/20

As the induction hypothesis, for $m' = 1, 2, \ldots, m$ with some fixed $m \in \{1, 2, \ldots, N-1\}$,

(i) $G_i^{(m')} > 0$ is equivalent to $H_i^{(m')} > 0$ for every $i \in \mathcal{N}$. In particular, if $q_j = 0$ for some $j \in \{i+1, \ldots, N\}$, then $G_i^{(m')} \leq 0$, and if $q_j > 0$ for all $j = i+1, \ldots, N$, then it holds that $G_i^{(m')} = (\prod_{j=i+1}^{N} q_j)\, H_i^{(m')}$.

(ii) $i \mapsto H_i^{(m')}$ changes sign from nonpositive to positive at most once.

(iii) $H_i^{(m'+1)} - H_i^{(m')} \geq 0$ for $i \in \mathcal{N}$ such that $H_i^{(m')} > -\infty$.

By the induction hypothesis, $H_i^{(m)} > 0$ and equivalently $G_i^{(m)} > 0$ for $i \geq i_*^{(m)}$. Thus, by (i) above, $q_j > 0$ for all $j = i_*^{(m)} + 1, \ldots, N$. Let us show (i)–(iii) above for $m' = m + 1$.

Proof of Multiple Sum-the-Odds Theorem 4/7 — 12/20

We first examine (i). From (2.11), the monotone selection region in the case with $m+1$ selection chances is given by $B^{(m+1)} = \{i \in \mathcal{N} : G_i^{(m+1)} > 0\}$, where

$$\begin{aligned} G_i^{(m+1)} &= V_i^{(m+1)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, V_j^{(m+1)}, \\ &= V_i^{(1)} + W_i^{(m)} - \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, (V_j^{(1)} + W_j^{(m)}) \\ &= G_i^{(1)} + \sum_{j=i+1}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, (M_j^{(m)} - W_j^{(m)}), \end{aligned} \tag{2.12}$$

where the first term on the right-hand side is obtained from the definition of $G_i^{(1)}$ and the second term is obtained from (2.3).

Proof of Multiple Sum-the-Odds Theorem 5/7 — 13/20

By the induction hypothesis, we have $M_j^{(m)} = V_j^{(m)}$ for $j \geq i_*^{(m)}$ and $M_j^{(m)} = W_j^{(m)}$ for $j < i_*^{(m)}$ in (2.1); that is,

$$M_j^{(m)} - W_j^{(m)} = \begin{cases} V_j^{(m)} - W_j^{(m)} & \text{for } j \geq i_*^{(m)}, \\ 0 & \text{for } j < i_*^{(m)}. \end{cases}$$

Furthermore, the induction hypothesis reads (2.3) as

$$W_j^{(m)} = \sum_{\ell=j+1}^{N} \left(\prod_{k=j+1}^{\ell-1} q_k \right) p_\ell \, V_\ell^{(m)} \quad \text{for } j \geq i_*^{(m)}.$$

Therefore, from (2.11), we have

$$M_j^{(m)} - W_j^{(m)} = G_j^{(m)} \quad \text{for } j \geq i_*^{(m)},$$

substituting this in (2.12), we have

$$G_i^{(m+1)} = G_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} \left(\prod_{k=i+1}^{j-1} q_k \right) p_j \, G_j^{(m)}, \quad i \in \mathcal{N}. \tag{2.13}$$

Proof of Multiple Sum-the-Odds Theorem 6/7 — 14/20

By the induction hypothesis, applying $G_i^{(m')} = (\prod_{j=i+1}^{N} q_j)\, H_i^{(m')}$ for $m' = 1$ and $m' = m$ to (2.13), we obtain

$$G_i^{(m+1)} = \left(\prod_{j=i+1}^{N} q_j\right) H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} \left(\prod_{k=i+1}^{j-1} q_k\right) p_j \left(\prod_{\ell=j+1}^{N} q_\ell\right) H_j^{(m)}$$
$$= \prod_{j=i+1}^{N} q_j \left(H_i^{(1)} + \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j\, H_j^{(m)}\right),$$

so that (2.7) leads to

$$G_i^{(m+1)} = \left(\prod_{j=i+1}^{N} q_j\right) H_i^{(m+1)}. \tag{2.14}$$

Hence, we have (i) for $m' = m + 1$.

Proof of Multiple Sum-the-Odds Theorem 7/7 — 15/20

1. Proof of (ii) $\rightarrow$ Skip.
2. Finally, to prove (iii) for $m' = m + 1$, we use (2.7) and take the difference between $H_i^{(m+2)}$ and $H_i^{(m+1)}$; that is,

$$H_i^{(m+2)} - H_i^{(m+1)} = \sum_{j=(i+1)\vee i_*^{(m+1)}}^{N} r_j \, H_j^{(m+1)} - \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, H_j^{(m)}$$

$$\geq \sum_{j=(i+1)\vee i_*^{(m)}}^{N} r_j \, (H_j^{(m+1)} - H_j^{(m)}) \geq 0,$$

where the first inequality follows from $H_j^{(m+1)} > 0$ for $j \geq i_*^{(m+1)}$ and $i_*^{(m+1)} \leq i_*^{(m)}$ by the induction hypothesis. The second inequality also follows from the induction hypothesis. Hence, the induction is completed and so is the proof.

Multiple Sum-the-Odds Theorem — 16/20

Let $h_i^{(m)} := 1 - H_i^{(m)}$ for i and $m \in \mathcal{N}$. We can observe that each $h_i^{(m)}$ is expressed as a combination of multiple odds-sums. For instance, $h_i^{(2)}$ and $h_i^{(3)}$ are calculated as

$$h_i^{(2)} = \sum_{j=i+1}^{i_*^{(1)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(1)}}^{N} r_j \sum_{k=j+1}^{N} r_k, \tag{2.15}$$

$$h_i^{(3)} = \sum_{j=i+1}^{i_*^{(2)}-1} r_j + \sum_{j=(i+1)\vee i_*^{(2)}}^{N} r_j \left\{ \sum_{k=j+1}^{i_*^{(1)}-1} r_k + \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k \sum_{\ell=k+1}^{N} r_\ell \right\}.$$

The optimal rule for the problem with m $(\in \mathcal{N})$ selection chances then reduces to $\tau_*^{(m)} = \min\{i \in \mathcal{N} : h_i^{(m)} < 1 \ \& \ X_i = 1\}$. Hence, we call this Theorem "multiple sum-the-odds theorem" or "multiple odds theorem" in short.

Probability of win — 17/20

For the problem with at most m ($\in \mathcal{N}$) selection chances, the maximum probability of win under the optimal rule, $P^{(m)}(\mathrm{win}) = P_N^{(m)}(p_1, \ldots, p_N)$, is given by

$$P^{(m)}(\mathrm{win}) = W^{(m)}_{i_*^{(m)}-1} = \prod_{j=i_*^{(m)}}^{N} q_j \sum_{j=i_*^{(m)}}^{N} r_j + \sum_{j=i_*^{(m)}}^{N} \left(\prod_{k=i_*^{(m)}}^{j} q_k \right) r_j \, W_j^{(m-1)}, \quad (3.1)$$

where if $p_{i_*^{(m)}} = 1$, then $P^{(m)}(\mathrm{win}) = \prod_{k=i_*^{(m)}+1}^{N} q_k + W^{(m-1)}_{i_*^{(m)}}$ (note that $p_j < 1$ for all $j = i_*^{(m)} + 1, \ldots, N$).

Probability of win for $m = 2$ and 3 — 18/20

1. Especially, for $m = 2$

$$P^{(2)}(\mathrm{win}) = \prod_{j=i_*^{(2)}}^{N} q_j \sum_{j=i_*^{(2)}}^{N} r_j \left(1 + \prod_{k=j+1}^{i_*^{(1)}-1} (1+r_k) \sum_{k=(j+1)\vee i_*^{(1)}}^{N} r_k\right). \qquad (3.2)$$

2. For $m = 3$,

$$P^{(3)}(\mathrm{win}) = \prod_{j=i_*^{(3)}}^{N} q_j \sum_{j=i_*^{(3)}}^{N} r_j \left[1 + \prod_{k=j+1}^{i_*^{(2)}-1} (1+r_k)\right.$$

$$\left. \times \sum_{k=(j+1)\vee i_*^{(2)}}^{N} r_k \left(1 + \prod_{\ell=k+1}^{i_*^{(1)}-1} (1+r_\ell) \sum_{\ell=(k+1)\vee i_*^{(1)}}^{N} r_\ell\right)\right]. \qquad (3.3)$$

Limiting probability of win — 19/20

1. In the following, to emphasize the dependence on N, we subscript "N" and write $P_N^{(m)}(\text{win})$ and $i_{*,N}^{(m)}$ occasionally. Let $R_N^{(m)} = \sum_{j=i_{*,N}^{(m)}}^N r_j$ and $R_N^{(m,2)} = \sum_{j=i_{*,N}^{(m)}}^N r_j^2$ for $m \in \mathcal{N}$.
2. For the maximum probability of win with $m = 2$, we have

$$P_N^{(2)}(\text{win}) > R_N^{(1)}\, e^{-R_N^{(1)}} + e^{-R_N^{(2)}}, \tag{3.4}$$

$$P_N^{(2)}(\text{win}) < R_N^{(1)}\, e^{-R_N^{(1)}+R_N^{(1,2)}} + (1 + r_{i_*^{(1)}}\, R_N^{(1)} + r_{i_*^{(2)}})\, e^{-R_N^{(2)}+R_N^{(2,2)}}. \tag{3.5}$$

Furthermore, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$, $R_N^{(1,2)} \to 0$ and $R_N^{(2,2)} \to 0$ as $N \to \infty$, then

$$P_N^{(2)}(\text{win}) \to e^{-1} + e^{-3/2} \quad \text{as } N \to \infty. \tag{3.6}$$

Conjectures on the probability of win — 20/20

We conjecture that, if $R_N^{(m)}$ and $R_N^{(m,2)}$, $m = 1, 2, \ldots$, have the same limits as those for the CSP with multiple selection chances, then the limit of the maximum probability of win is also consistent with that for the CSP; that is,

$$\lim_{N\to\infty} P_N^{(m)}(\mathrm{win}) = \lim_{N\to\infty} \sum_{j=1}^{m} \frac{i_*^{(j)}}{N} \quad \text{for } m = 1, 2, \ldots.$$

For instance, for the triple selection problem, our conjecture states that, if $R_N^{(1)} \to 1$, $R_N^{(2)} \to 3/2$ and $R_N^{(3)} \to 47/24$ with $R_N^{(m,2)} \to 0$, $m = 1, 2, 3$ as $N \to \infty$, then

$$\lim_{N\to\infty} P_N^{(3)}(\mathrm{win}) = e^{-1} + e^{-3/2} + e^{-47/24}.$$

Selected references

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