Solutions for bidding games with two risky assets: general case

Victor DOMANSKY, Victoria KREPS

St.Petersburg Inst. for Econ. & Math., Russian Academy of Sciences

e-mail: doman@emi.nw.ru

In the previous report we give solutions for such games with two and three possible values of vector of liquidation prices. Here we extend these solutions to the games $G(\mathbf{p})$ with arbitrary distributions $\mathbf{p} \in \Delta(\mathbb{Z}^2)$ with finite variances.

To define the optimal strategy of Player 1 we construct the symmetric representation of probability distributions $\mathbf{p} \in \Theta(k,l)$ over the two-dimensional integer lattice with a fixed integer expectation vector (k,l) as convex combinations of extreme points of the set $\Theta(k,l)$, i.e. distributions with not more than three-point supports. This is sufficient to give such representation for the set $\Theta(0,0)$.

This representation is a straight generalization of the analogous representation for distributions $\mathbf{p} \in \Theta(k)$ over one-dimensional integer lattice with a fixed integer expectation, exploited in Domansky and Kreps (2009). Consider the set

$$\Theta(0) = \{\mathbf{p} \in \Delta(\mathbb{Z}^1) : \mathbf{E}_{\mathbf{p}}[u] = 0\}.$$

The extreme points of the set $\Theta(0)$ are:
1) The degenerate distribution $\mathbf{e}^0$ with one-point support $\{0\}$;
2) distributions $\mathbf{p}^0_{k,l} \in \Theta(0)$ with two-point supports $\{-l, k\}$,

$$p^0_{k,l}(-l) = \frac{k}{k+l}, \quad p^0_{k,l}(k) = \frac{l}{k+l}.$$

Any distribution $\mathbf{p} \in \Theta(0)$ has the following symmetric representation:

$$\mathbf{p} = p(0) \cdot \mathbf{e}^0 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \alpha_{k,l}(\mathbf{p}) \cdot \mathbf{p}^0_{k,l}$$

with the coefficients

$$\alpha_{k,l}(\mathbf{p}) = \frac{k+l}{\sum_{t=1}^{\infty} t \cdot p(t)} p(-l)p(k).$$

Here we give an analogous symmetric representation of probability distributions $\mathbf{p} \in \Theta(0,0)$, where

$$\Theta(0,0) = \{\mathbf{p} \in \Delta(\mathbb{Z}^2) : \mathbf{E}_\mathbf{p}[u] = 0, \mathbf{E}_\mathbf{p}[v] = 0\},$$

as convex combinations of extreme points of the set $\Theta(0,0)$, i.e. one-, two-, and three-point distributions.

a) Put $\mathbf{p}^1 = p(0,0)\mathbf{e}^\mathbf{0}$, where $\mathbf{e}^\mathbf{0} \in \Theta(0,0)$ is the degenerate distribution. This is one-point part of distribution $\mathbf{p} \in \Theta(0,0)$.

b) The support of two-point distribution $\mathbf{p} \in \Theta(0,0)$ is situated over a straight line passing through $(0,0)$. Such line is uniquely defined with a point $w = (u,v) \in \mathbb{Z}^2, v \geq 0$ on it, with (u,v) being relatively prime. Let $W \subset \mathbb{Z}^2$ be the set of such points (including $(1,0)$ but not $(-1,0)$).

Let $\mathbf{p}^0_{wk,-wl} \in \Theta(0,0)$, $k,l = 1,2,\ldots,$ be the distribution with the two-point support $\{-l \cdot w, k \cdot w\}$.

Put $m^-(\mathbf{p}_w) = \sum_{k=1}^{\infty} k \cdot p(-k \cdot w)$,

$$m^+(\mathbf{p}_w) = \sum_{k=1}^{\infty} k \cdot p(k \cdot w).$$

Here $m(\mathbf{p}_w) = -m^-(\mathbf{p}_w) + m^+(\mathbf{p}_w)$ is the central moment of the part $\mathbf{p}_w$ of distribution $\mathbf{p} \in \Theta(0,0)$, lying on the straight line passing through $(0,0)$ and w.

For any $k \in \mathbb{N}, w \in W$, put

$$p^2(-k \cdot w) = \frac{p(-k \cdot w) \cdot m^+(\mathbf{p}_w)}{m^-(\mathbf{p}_w) \vee m^+(\mathbf{p}_w)},$$

$$p^2(k \cdot w) = \frac{p(k \cdot w) \cdot m^-(\mathbf{p}_w)}{m^-(\mathbf{p}_w) \vee m^+(\mathbf{p}_w)},$$

where $m_w^-(\mathbf{p}) \vee m_w^+(\mathbf{p}) = \max(m_w^-(\mathbf{p}), m_w^+(\mathbf{p}))$.

The substochastic distribution $\mathbf{p}^2 \in P^2$, where P^2 is the class of distributions, such that the moment on any straight line passing through $(0,0)$ is equal to zero. They are represented as combinations of two-point distributions.

Proposition 1. *The part* $\mathbf{p}^2$ *of distribution* $\mathbf{p} \in \Theta(0,0)$ *has the following representation as a combination of two-point distributions:*

$$\mathbf{p}^2 = \sum_{w \in W} \sum_{k,l=1}^{\infty} \alpha_{wk,-wl}(\mathbf{p}) \cdot \mathbf{p}^0_{wk,-wl}$$

with the coefficients

$$\alpha_{wk,-wl}(\mathbf{p}) = \frac{k+l}{\sum_{t=1}^{\infty} t \cdot p^2(t \cdot w)} p^2(-l \cdot w) p^2(k \cdot w)$$

$$= \frac{k+l}{m_w^-(\mathbf{p}) \vee m_w^+(\mathbf{p})} p(-l \cdot w) p(k \cdot w).$$

c) The substochastic distribution

$$\mathbf{p}^3 = \mathbf{p} - \mathbf{p}^1 - \mathbf{p}^2 \in P^3,$$

where P^3 is the class of distributions, such that, for any straight line passing through $(0,0)$, they have only one loaded half-line. They are represented as combinations of three-point distributions.

Let three points $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ can be enumerated so that

$$\det[z_i, z_{i+1}] > 0, \quad i = 1, 2, 3, \qquad (1)$$

where $\det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$. All arithmetical operations with subscripts are fulfilled modulo 3. Then there is a unique distribution $\mathbf{p}^0_{z_1,z_2,z_3} \in \Theta^3(0,0)$ with three-point support (z_1, z_2, z_3), namely

$$p^0_{z_1,z_2,z_3}(z_i) = \frac{\det[z_{i+1}, z_{i+2}]}{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}.$$

Let Δ^0 be the set of three-point sets (z_1, z_2, z_3) satisfying (1).

Let $\Delta^0(z)$ be the set of ordered pairs (z_2, z_3), such that $\det[z_2, z_3] > 0$ and the set (z, z_2, z_3) belongs to Δ^0,

Lemma. *For any distribution* $\mathbf{p} \in P^3$

$$\sum_{(z_2,z_3)\in\Delta^0(z)} p(z_2)p(z_3)\det[z_2, z_3] = \Phi(\mathbf{p}),$$

does not depend on z, *i.e. this is an invariant of the distribution* $\mathbf{p}$.

This is a two-dimensional analog of the fact that for $\mathbf{p} \in \Theta(0) \subset \Delta(\mathbb{Z}^1)$ the equality $\sum_{t=1}^{\infty} t \cdot p(t) = \sum_{t=1}^{\infty} t \cdot p(-t)$ holds.

Proposition 2. *The part* $\mathbf{p}^3$ *of distribution* $\mathbf{p} \in \Theta(0, 0)$ *has the following representation as a combination of three-point distributions:*

$$\mathbf{p}^3 = \sum_{\Delta^0} \alpha_{z_1,z_2,z_3}(\mathbf{p}) \cdot \mathbf{p}^0_{z_1,z_2,z_3}$$

with the coefficients

$$\alpha_{z_1,z_2,z_3}(\mathbf{p}) = \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(\mathbf{p}^3)} p^3(z_1)p^3(z_2)p^3(z_3).$$

Corollary. *Any linear function f over the set $\Theta(0,0)$ has the following representation as a convex combination of its values over the extreme points of $\Theta(0,0)$:*

$$f(\mathbf{p}) = p(0,0) f(\mathbf{e^0}) + \sum_{w \in W} \sum_{k,l=1}^{\infty} \alpha_{wk,-wl}(\mathbf{p}) \cdot f(\mathbf{p}^0_{wk,-wl})$$

$$+ \sum_{(z_1,z_2,z_3) \in \Delta^0} \alpha_{z_1,z_2,z_3}(\mathbf{p}) \cdot f(\mathbf{p}^0_{z_1,z_2,z_3})$$

with the coefficients $\alpha_{wk,-wl}(\mathbf{p})$ and $\alpha_{z_1,z_2,z_3}(\mathbf{p})$ given by Propositions 1 and 2.

Now we construct optimal strategies for Player 1 making use of the constructed decomposition for the initial distribution $\mathbf{p}$.

a) If the state chosen by chance move is $(0,0)$, then Player 1 stops the game.

Let the state chosen by chance move be $z = k \cdot w$, where $k \in \mathbb{Z}$, $k \neq 0$, and $w \in W$. For definiteness let be $k > 0$.

b) If $m^{+}(\mathbf{p}_w) \leq m^{-}(\mathbf{p}_w)$, then the state z belongs to the support of the distribution $\mathbf{p}^2$ and does not belong to the support of the distribution $\mathbf{p}^3$. Player 1 chooses a point $z_2 = -l \cdot w$ by means of lottery with probabilities

$$q(-l \cdot w) = \frac{l \cdot p(-l \cdot w)}{m^{-}(\mathbf{p}_w)},$$

and plays the optimal strategy $\sigma^{*}(\cdot|z)$ for the state z in the two-point game $G(\mathbf{p}^{0}_{wk,-wl})$.

c) Otherwise, if $m^+(\mathbf{p}_w) > m^-(\mathbf{p}_w)$, then the state z belongs to the support of the both distributions $\mathbf{p}^2$ and $\mathbf{p}^3$ with probabilities

$$\frac{m^-(\mathbf{p}_w)}{m^+(\mathbf{p}_w)} \quad \text{and} \quad 1 - \frac{m^-(\mathbf{p}_w)}{m^+(\mathbf{p}_w)}$$

correspondingly. Player 1 chooses a distributions $\mathbf{p}^2$ or $\mathbf{p}^3$ by means of lottery with these probabilities.

d) If the distribution $\mathbf{p}^2$ is chosen, then further Player 1 acts as in the point b).

e) If the distribution $\mathbf{p}^3$ is chosen, then Player 1 chooses a pair $(z_2, z_3) \in \Delta^0(w)$ by means of lottery with probabilities

$$q(z_2, z_3) = \frac{p^3(z_2)p^3(z_3) \cdot \det[z_2, z_3]}{\Phi(\mathbf{p}^3)},$$

and plays the optimal strategy $\sigma^*(\cdot|z)$ for the state z in the three-point game $G(\mathbf{p}^0_{z,z_2,z_3})$.

As the optimal strategies σ^* ensure Player 1 the gains equal to one half of the sum of component variances $\mathbf{D}_{\mathbf{p}}[u] + \mathbf{D}_{\mathbf{p}}[v]$ in the two and three-point games and as the sum of component variances is a linear function over $\Theta(0,0) \cap M^2$, where M^2 is the class of distributions with finite second moment, we obtain the following result:

Theorem. *For any distribution* $\mathbf{p} \in M^2$ *the compound strategy depicted above ensures Player 1 the gain* $1/2 \cdot (\mathbf{D}_{\mathbf{p}}[u] + \mathbf{D}_{\mathbf{p}}[v])$ *in the game* $G(\mathbf{p})$.