American options with guarantee

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Outline

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Problem

Options with guarantee

Guarantee a payoff that is a fraction of the starting value

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$$g(X_\tau) \vee h(X_0)$$

g: gain function, h: guarantee function
g, h increasing, $h \leq g$

Optimal stopping problem

$$v(x) = \sup_{\tau} E_x(e^{-r\tau}\, g(X_\tau) \qquad 1_{\{\tau<\infty\}}), \quad x \in E$$

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- Structure of the optimal strategies?
- Explicit determination?

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Let ψ_+, ψ_- be the *minimal r-harmonic functions*, i.e.

$$\psi_+(x) = \begin{cases} E_x(e^{-r\tau_a} 1_{\{\tau_a < \infty\}}), & x \leq a \\ [E_a(e^{-r\tau_x} 1_{\{\tau_x < \infty\}})]^{-1}, & x > a \end{cases}$$

and

$$\psi_-(x) = \begin{cases} [E_a(e^{-r\tau_x} 1_{\{\tau_x < \infty\}})]^{-1}, & x \leq a \\ E_x(e^{-r\tau_a} 1_{\{\tau_a < \infty\}}), & x > a \end{cases}$$

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for a fixed point $a \in int(I)$.
ψ_+ resp. ψ_- are the (up to a factor) unique incresing resp. decreasing solutions to

$$(A - r)\psi = 0$$

and each solution ψ can be written as $\psi = \lambda_1 \psi_+ + \lambda_2 \psi_-$ for some $\lambda_1, \lambda_2 \geq 0$.

One-dimensional diffusions

Basic tool for problems without guarantee (C., I. 2010):

The optimal stopping set for

$$v(x) = \sup_{\tau} E_x(e^{-r\tau}\tilde{g}(X_\tau)1_{\{\tau<\infty\}})$$

is given by the maximum points of

$$\frac{\tilde{g}}{\lambda\psi_+ + (1-\lambda)\psi_-}, \qquad \lambda \in [0,1].$$

One-dimensional diffusions

Corollary

Let $x \in I$ and assume there exist $y_1 \leq x \leq y_2$ and $\lambda_1, \lambda_2 \in [0,1]$ such that

$$y_i = argmax \frac{\tilde{g}}{\lambda_i \psi_+ + (1-\lambda_i)\psi_-} \qquad \text{for } i = 1, 2.$$

Then there exist $x_1 \leq x \leq x_2$ such that $v(x) = E_x(e^{-r\tau}\tilde{g}(X_\tau))$, where

$$\tau = \inf\{t \geq 0 : X_t \leq x_1 \text{ or } X_t \geq x_2\}.$$

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Proof (Corollary):
By the basic tool y_1 and y_2 are in the stopping set S. Hence

$$x_1 := \sup\{y \in S : y \leq x\} \quad \text{and} \quad x_2 := \inf\{y \in S : y \geq x\}$$

are in S too, i.e. $\tau_S = \tau$ under P_x and since the interval $[x_1, x_2]$ is compact the assertion holds.

One-dimensional diffusions

Theorem

Consider the optimal stopping problem with guarantee

$$v(x) = \sup_{\tau} E_x(e^{-r\tau}[g(X_\tau) \vee h(X_0)]1_{\{\tau<\infty\}}), \quad x \in E.$$

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Theorem

Consider the optimal stopping problem with guarantee

$$v(x) = \sup_{\tau} E_x(e^{-r\tau}[g(X_\tau) \vee h(X_0)]1_{\{\tau<\infty\}}), \quad x \in E.$$

Assume $\lim_{y \to b_r} \frac{g(y)}{\psi_+(y)} = 0$.
For each starting point x with $h(x) > 0$ there exist $a_x \leq x \leq b_x$ such that

$$\tau_x^* = \inf\{t \geq 0 : X_t = a_x \text{ or } X_t = b_x\}$$

is optimal.

Proof of the Theorem

It is easy to see that for λ near 1

$$\sup_{y\leq x}\frac{g(y)\vee h(x)}{\lambda\psi_+(y)+(1-\lambda)\psi_-(y)} > \sup_{y\geq x}\frac{g(y)\vee h(x)}{\lambda\psi_+(y)+(1-\lambda)\psi_-(y)}$$

Therefore by assumption on the boundary behavior there exist $\tilde{a}_x \leq x$ and λ_1 such that

$$\tilde{a}_x = argmax_y \frac{g(y)\vee h(x)}{\lambda_1\psi_+(y)+(1-\lambda_1)\psi_-(y)}.$$

The same argument for λ near 0 provides $\tilde{b}_x \geq x$ with

$$\tilde{b}_x = argmax_y \frac{g(y)\vee h(x)}{\lambda_2\psi_+(y)+(1-\lambda_2)\psi_-(y)}.$$

The assertions follows from the Corollary. □

Explicit determination of the optimal boundaries

Assume $g(x) = h(x)$. Write

$$F(x, a, b) := E_x(e^{-r\tau_{a,b}}[g(X_{\tau_{a,b}}) \vee g(x)]1_{\{\tau_{a,b}<\infty\}}),$$

where $\tau_{a,b} = \inf\{t \geq 0 : X_t = a \text{ or } X_t = b\}$.
(a_x, b_x) is maximum point of $F(x, \cdot)$.

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Reduction to ODE

$$\frac{d}{dx}(a_x, b_x) = -(D^2_{2,3}F(x,a_x,b_x))^{-1}D_1D_{2,3}F(x,a_x,b_x).$$

Example: GBM-Stock with guarantee

Black-Scholes market: X is GBM, $g(x) = h(x) = x$.

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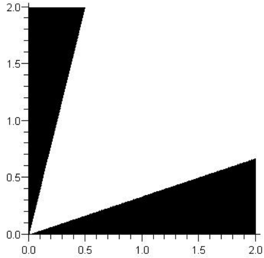
$$\tau_x^* = \inf\{t \geq 0 : ax \leq X_t \text{ or } X_t \leq bx\},$$

$$v(x) = cx.$$



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Lévy processes

Now let X be a Lévy process.

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Proposition

(i) For all $x \in \mathbb{R}$ there exists $a_x \in [-\infty, x]$ such that

$$S_x \cap (-\infty, x] = (-\infty, a_x].$$

(ii) For all $x \in \mathbb{R}$ it holds that

$$S_x \cap [x, \infty) \neq \emptyset.$$

Spectrally negative Lévy processes

For analytic tractability: X is spectrally negative, i.e. it has no upward jumps, $h = g$.

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Theorem

For each starting point x with $h(x) > 0$ there exist $-\infty < a_x \leq x \leq b_x < \infty$ such that

$$\tau_x^* = \inf\{t \geq 0 : X_t \leq a_x \text{ or } X_t = b_x\}$$

is optimal.

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Thank you for your attention!