Solutions for bidding games with two risky assets: the case of two and three states

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Bachelier L. (1900) *Theorie de speculation.* Ann. Ecole Norm. Sup., 1900, 17, 21-86.

Random walks (brownian motion) for describing the evolution of prices on stock markets (see A.N.Shirjaev, 1998)

Statistical analysis of time series of prices at stock markets discovers regular random fluctuations.

Endogenious explanation:
Kyle A.S. (1985) *Continuous auctions and insider trading.* Econometrica, V.53, 1315-1335.

The random fluctuations in the evolution of prices on the stock markets may originate from asymmetric information of stockbrokers on events determining market prices.

De Meyer B., Moussa Saley H. (2002) *On the Strategic Origin of Brownian Motion in Finance.* Int. Journal of Game Theory, 31, 285-319.

The MODEL

The bidding between two agents for single-type risky assets (shares)

TWO PLAYERS have
MONEY + SHARES of one type
RANDOM liquidation price of a share
may take any non-negative
INTEGER value.

STEP 0: ONCE FOR ALL
a chance move determines
a liquidation price of one share
according to a probability distribution
$\mathbf{p} = (p_0, p_1, p_2, \ldots)$, $\mathbf{p} \in \Delta(\mathbb{Z}^1)$.

Both players know distribution $\mathbf{p}$.

Player 1 (insider) is informed
on the chosen price.

Player 2 is not.

Player 2 knows that Player 1 is an insider.

STEP t, $t = 1, 2, \ldots, n$:
Players simultaneously propose
their prices for one share,
i_t for Player 1,
j_t for Player 2.

Any integer bids are admissible.
The pair (i_t, j_t) is announced to both Players
before proceeding to the next stage.

The player who posts the LARGER price
buys one share from his opponent
for THIS price.

Players aim to maximize the values of their
final portfolios, i.e. money plus liquidation values
of obtained shares.

The model is described by a zero-sum n-stage repeated game $G_n(\mathbf{p})$ with incomplete info of Player 2 (Aumann, Maschler (1995)).

One-step payoffs of Player 1 is given by

$$a^s(i,j) = \begin{cases} j - s, & \text{if } i < j; \\ 0, & \text{if } i = j; \\ -i + s, & \text{if } i > j, \end{cases}$$

where s is the result of chance move.

The final payoff is $\sum_{t=1}^{n} a^s(i_t, j_t)$.

If the expectation $\mathbf{E}[\mathbf{p}] < \infty$,
then the game $G_n(\mathbf{p})$ has a value $V_n(\mathbf{p})$.

If the variance $\mathbf{D}[\mathbf{p}] < \infty$, then the sequence $V_n(\mathbf{p})$ is bounded from above by a continuous concave piecewise linear function $H(\mathbf{p})$.
Its domains of linearity are

$$L(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] \in [k, k+1]\}, \quad k = 0, 1, \ldots.$$

If $\mathbf{E}[\mathbf{p}]$ is an integer, then $H(\mathbf{p}) = 1/2 \cdot \mathbf{D}[\mathbf{p}]$.

$H(\mathbf{p})$ is the exact upper bound,

$$\lim_{n\to\infty} V_n(\mathbf{p}) = H(\mathbf{p}).$$

We may consider the game $G_\infty(\mathbf{p})$ with UNLIMITED BEFOREHAND number of steps.

We solve the game $G_\infty(\mathbf{p})$.

Its value $V_\infty(\mathbf{p}) = H(\mathbf{p})$.

The OPTIMAL strategy of Player 2:
his *first* move is the action k, where k is the integer part of expectation of share price;

his move *at stage t*, $t > 1$ depends on the last observed pair of actions (i_{t-1}, j_{t-1}) only:

$$j_t = \begin{cases} j_{t-1} - 1, & \text{for } i_{t-1} < j_{t-1} \ ; \\ j_{t-1}, & \text{for } i_{t-1} = j_{t-1}; \\ j_{t-1} + 1, & \text{for } i_{t-1} > j_{t-1}. \end{cases}$$

Let the initial expectation
of share price be equal to integer k.

The OPTIMAL strategy of Player 1:
Stage 1. Chosen price s may be equal to k.
It occurs with probability p_k.
If so, Player 1 stops the game.

Otherwise, his first move makes use of
two actions $k-1$ and k
with total probabilities 1/2 such that
the price expectation

after stage 1 $k-1$, or $k+1$;

Stage 2. If after stage 1
the price expectation becomes equal to s,
Player 1 stops the game.

Otherwise,

after stage 2 $k-2$ or k, k or $k+2$;

and so on ...

The optimal strategy of Player 1 generates an ELEMENTARY SYMMETRIC RANDOM WALK of price expectation with absorption. The ABSORPTION means REVEALING the REAL VALUE of SHARE by Player 2.

The expected duration of this symmetric random walk before absorption is equal to the variance $\mathbf{D}[\mathbf{p}]$ of the liquidation price of a share.

The value of game $G_\infty(\mathbf{p})$ is equal to the expected duration of this random walk, multiplied by the constant one-step expected gain 1/2 of Player 1.

The random sequence of prices of realized transactions represents the SAME RANDOM WALK.

Domansky V., Kreps V. (2009) *Repeated games with asymmetric information and random price fluctuations at finance markets: the case of countable state space*. Centre d'Economie de la Sorbonne. Univ. Paris 1. Pantheon – Sorbonne. Preprint 2009.40.

1) **Particular case**: $\mathbf{p}$ has TWO-point support, $p_0 = 1 - p, \quad p_m = p$.

The OPTIMAL strategy of Pl.1 generates an ELEMENTARY SYMMETRIC RANDOM WALK of POSTERIOR probabilities of high share price over the points

$$\frac{l}{m}, \quad l = 0, \ldots, m,$$

with absorbing extreme points 0 and 1.

This OPTIMAL strategy of Pl.1 is not unique.
Pl.1 may generate a SLOWER
both symmetric and asymmetric
random walk of posterior probabilities
with THE SAME TOTAL GAIN.

2) **General case.**
The unique "canonical" decomposition of $\mathbf{p} \in \Delta(\mathbb{Z}^1)$ with integer expectation as a convex combination of TWO-point support distributions with the same expectation.

Constructing the optimal strategy of Player 1 for the game $G_\infty(\mathbf{p})$ as the convex combination of his optimal strategies for games with two-point support distributions with coefficients of decomposition of distribution $\mathbf{p}$.

The bidding between two agents for shares of TWO types

TWO PLAYERS have
MONEY + SHARES of two types
RANDOM liquidation prices
may take any integer values.

STEP 0: a chance move determines
a pair $(u, v) \in \mathbb{Z}^2$ of liquidation prices
according to a probab. distrib. $\mathbf{p} \in \Delta(\mathbb{Z}^2)$
ONCE FOR ALL.

Both players know distribution $\mathbf{p}$.

Player 1 (insider) is informed
on the chosen prices u, v.

Player 2 is not.

Player 2 knows that Player 1 is an insider.

STEP t, $t = 1, 2, \ldots, n$:
Players simultaneously propose their prices for each type of shares,
$(i_t(1), i_t(2)) \in \mathbb{Z}^2$ for Player 1,
$(j_t(1), j_t(2)) \in \mathbb{Z}^2$ for Player 2.

Any integer vector bids are admissible.

The player who posts the larger price for a share of given type buys one share of this type from his opponent for this price.

Game $G_n(\mathbf{p})$ with two-dimensional one-step actions.

If the expectations of share prices are finite, then the value $V_n(\mathbf{p})$ exists and

$$V_n(\mathbf{p}) \leq V_n(\mathbf{p}^1) + V_n(\mathbf{p}^2),$$

$\mathbf{p}^1$, $\mathbf{p}^2$ are one-dimensional marginals of $\mathbf{p}$.

If share prices have finite variances, then

$$V_n(\mathbf{p}) \leq 1/2 \cdot (\mathbf{D}[\mathbf{p}^1] + \mathbf{D}[\mathbf{p}^2]).$$

The bidding of unbounded beforehand duration and the games $G_\infty(\mathbf{p})$, $\mathbf{p} \in \mathbf{\Delta}(\mathbb{Z}^{\mathbf{2}})$.

The Result:

$$V_\infty(\mathbf{p}) = V_\infty(\mathbf{p}^1) + V_\infty(\mathbf{p}^2),$$

The optimal strategies for both players.

Particular cases:

a) $\mathbf{p}$ has TWO-point support;

b) $\mathbf{p}$ has THREE-point support.

a) $\mathbf{p}$ has TWO-point support;
$(0, 0)$ and (m_1, m_2), where $m_1 \neq m_2 > 0$.

The optimal strategy $\sigma^*(p)$ of Player 1?

Consider the lattice $D(m_1, m_2)$,

$\{k/m_1, k = 0, \ldots, m_1\} \cup \{l/m_2, l = 0, \ldots, m_2\}$.

Let $p \in D(m_1, m_2)$.
For $l/m_2 < p = k/m_1 < (l+1)/m_2$, the first move of the strategy $\sigma^*(p)$ makes use of two actions $(k-1, l)$ and (k, l).
For $k/m_1 < p = l/m_2 < (k+1)/m_1$, it uses actions $(k, l-1)$ and (k, l).
For $p = k/m_1 = l/m_2$, it uses actions $(k-1, l-1)$ and (k, l).

The posterior probabilities of state (m_1, m_2) are the left and the right adjacent points of the lattice $D(m_1, m_2)$.

The total probabilities of actions provide the martingale characteristics of posterior probabilities.

The optimal strategy of Player 1 generates a NON-SYMMETRIC RANDOM WALK of prices over $\mathbb{Z}^2$.

b) The support of initial distribution contains 3 points $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$, $z_1, z_2, z_3 \in \mathbb{Z}^2$.

Such distribution is uniquely determined with expectations of coordinates.

For any point $w = (u, v) \in \triangle(z_1, z_2, z_3)$ the distribution $\mathbf{p}_w$ such that

$$\mathbf{E}_{\mathbf{p}_w}[x] = u \ and \ \mathbf{E}_{\mathbf{p}_w}[y] = v$$

is given with probabilities

$$p_w(z_i) = \frac{\det[z_{i+1} - w, z_{i+2} - w]}{\det[z_1 - z_3, z_2 - z_3]}, \quad i = 1, 2, 3,$$

where $\det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$.

Notice that arithmetical operations with subscripts are fulfilled modulo 3.

Let $w = (u, v) \in \mathbb{Z}^2$ be integer.
Denote $e = (1, 1), \bar{e} = (1, -1)$.

Two types of optimal first moves of Player 1
NE-SW and NW-SE.

The first moves $\sigma_1^{NE-SW}(\mathbf{p}_w)$ makes use of two actions $(u-1, v-1)$ and (u, v) with posterior expectations $w - b \cdot e$ and $w + a \cdot e$.

If $w + e \in \triangle(z_1, z_2, z_3)$, then $a = 1$. Otherwise $w + a \cdot e$ is a boundary point of the triangle $\triangle(z_1, z_2, z_3)$.

If $w - e \in \triangle(z_1, z_2, z_3)$, then $b = 1$.
Otherwise $w - b \cdot e$ is a boundary point of the triangle $\triangle(z_1, z_2, z_3)$.

The total probabilities of actions provide the martingale characteristics of posterior probabilities.

The martingale of posterior expectations
generated by
the optimal strategy of Player 1
is a symmetric random walk
over the adjacent points of the lattice $\mathbb{Z}^2$
disposed inside the triangle $\triangle(z_1, z_2, z_3)$.

The symmetry of this random walk is broken
at the moment when it hits the triangle boundary.
Beginning from this moment the game degenerates
into one of two-point games
with the distribution support being
either (z_1, z_2), or (z_2, z_3), or (z_3, z_1).