A new approach to the solution of optimal stopping problem

E.L.Presman

CEMI RAS, Russia
47 Nakhimovsky prospect, Moscow, 117428, Russia
E-mail: presman@cemi.rssi.ru

An optimal stopping problem of Markov chain with infinite horizon is considered. For the case of finite number m of states I.M.Sonin proposed an algorithm, which allows to find the value function and the stopping set in no more than $2(m-1)$ steps. The algorithm is based on a **modification of Markov chain on each step**, related with elimination of the states which certainly belong to the continuation set. To solve the problem with arbitrary state space and to have a possibility of generalization to the continuous time one needs to modify the procedure. We propose a procedure which is based on a sequential **modification of the payoff function** for **the same chain** in such a way, that the value function is the same for both problems and the modified payoff function is greater than the initial one on some set and is equal to it on the complement. We show the efficiency of this procedure and discuss the possibilities for generalizations in continuous time.

Key words: Markov chain, optimal stopping, elimination algorithm

1. Introduction

A general approach to find the value function in a problem of optimal stopping of a time homogeneous Markov chain with infinite horizon is as follows. One considers a recurrent sequence $\tilde{V}_k(x),\ k \geq 0$, where $\tilde{V}_k(x)$ is the value function in an optimal stopping problem on the time interval $[0, k]$. It is known that under standard assumptions the sequence $\tilde{V}_k(x)$ converges to the value function for the infinite horizon. It is often said that such approach gives a constructive method of finding the value function. Nevertheless, even for the case of Markov chain with finite number of states this procedure gives the exact value of the value function as a rule only after infinite number of steps.

For the case of finite number m of states I.M. Sonin (see [8, 9, 10, 11]) proposed an algorithm, which allows to find the value function and the stopping set in no more than $2(m-1)$ steps. The fact underlying this algorithm is the following. Those points where expected reward for doing one more step is larger than the expected reward from immediate stopping belong certainly to the continuation set. Therefore these points can be eliminated and we can consider a new chain, with new reduced state space and new transition probabilities. These probabilities coincide with the distribution of the initial chain at the time of the first return to the new state space. They can be simply recalculated from the old ones. In the case of finite number of states after finite number of steps we obtain the new chain and the new state space for which the reward for stopping — which equals to payoff function – is greater than or equal to the expected reward for doing one more step for all points. In such situation the stopping set coincides with the final state space and the value function coincides with the reward for instant stopping. After that the value functions corresponding to the previous chains can be restored sequentially. The possibilities of generalization to the countable case in some situations were discussed in [11].

The case of arbitrary state space was considered in [4]. Just as in [8, 9, 10, 11] the set C where the reward for the immediate stopping is less than the expected reward for continuation in one step is selected. Obviously in this case $C \subseteq C^*$, where C^* is the continuation set. By analogy with [8, 9, 10, 11] a new chain was defined but with the same state space and the same starting point z as for the initial one. The new chain coincides with the initial one at the times of the sequential returns of the latter chain to D in case $z \in D$ and with the times of sequential visits D in case $z \in C$, where D is the complement of C. It was proved that the sequence of sets $C_k, 1 \leq k \leq \infty$, obtained by the sequential repetition of this procedure is nondecreasing and converges to the continuation set of the initial chain. An application of this approach was demonstrated in [5] for a specific two-dimensional chain.

It turns out that for possibility to generalize the approach to continuous time it is reasonable to change the procedure. Instead of modifying the chain one needs on each step to modify the payoff function, changing it on the set C to the expected reward at the time of the first exit from C. The modified payoff function is greater than or equal to the initial one and the value function is the same for both problems. The increasing sequence of sets C_k remains the same as in [4] and the corresponding sequence of the modified payoff functions converges nondecreasingly to the value function of the initial problem.

In Section 2 we remind the well known facts from the theory of the optimal stopping of Markov chains with infinite horizon. In Section 3 we describe a proposed approach with the modification of the payoff function. In Section 4 the proof of the Main Lemma from Section 3 is given.

In Section 5 we demonstrate the possibilities of generalizations to the continuous time by an example of the geometric Brownian motion on the interval $[1, +\infty)$ with the reflection at the point 1 and by three other examples.

2. Optimal stopping problem

We consider a time homogeneous Markov chain $Z = (Z_n)_{n \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbf{P}_z)$ and taking values in a measurable space $(X, \mathcal{B})$. It is assumed that the chain Z starts at z under $\mathbf{P}_z$ for $z \in X$. It is also assumed that the mapping $z \mapsto \mathbf{P}_z(F)$ is measurable for each $F \in \mathcal{F}$. Denote by $\mathcal{P}$ the transition operator of Z, so that $E_z[f(Z_1)] = \mathcal{P}f(z)$ for any f, such that corresponding expectation exists.

A number β, $0 < \beta \leq 1$, and measurable payoff function $g(z)$ and cost function $c(z)$ are given. Stopping times are considered with respect to sequence of σ-algebras $\mathcal{F}_n$, $n \geq 0$. Here β is a discount coefficient, $g(z)$ is a reward from stopping at point z, and $c(z)$ is a fee for the observation (both functions can take positive and negative values). The problem of optimal stopping consists, first, in finding the value function

$$V(z) = \sup_\tau V^\tau(z), \text{ where } V^\tau(z) = E_z \left[g\left(Z_\tau\right) \beta^\tau - \sum_{k=0}^{\tau-1} c(Z_k) \beta^k \right], \tag{1}$$

and the supremum is taken over all stopping times, and, second, in finding an optimal stopping time, i.e. the stopping time where the supremum is achieved.

It is well known that the case $0 < \beta < 1$ can be reduced to the case $\beta = 1$ by introducing an absorbing state, which we shall denote by e. The probability of transition to e from any state of X is equal to $1-\beta$ and the new transition probabilities between states from X are equal to the old ones multiplied by β (see, for example, [4]). Then

$$E_z\left[g\left(Z_\tau\right)\beta^\tau - \sum_{k=0}^{\tau-1} c(Z_k)\beta^k\right] = \tilde{E}_z\left[g\left(Z_\tau\right) - \sum_{k=0}^{\tau-1} c(Z_k)\right],$$

where $\tilde{E}_z$ corresponds to the new transition probabilities.

Thus, in what follows we assume that $\beta = 1$.

Let us define an operator T as follows:

$$Tf(z) = -c(z) + \mathcal{P}f(z). \tag{2}$$

The operator T is called the reward operator.

It is well known that under natural assumptions (see, for example, [3], p. 12, condition (2.1.1)) the following statement holds (see [3], Theorem 1.11, Corollary 1.12 and Section 11 of Chapter 1; or [7], Section 14):

Theorem 1. a) *The value function $V(z)$ is the minimal solution of the Bellman (optimality) equation*

$$V(z) = \max[g(z), TV(z)]. \tag{3}$$

b) *If $\mathbf{P}_z[\tau^* < \infty] = 1$ for all $z \in X$, where $\tau^* = \inf\{n \geq 0 : \ Z_n \in D^*\}$ and $D^* = \{z : V(z) = g(z)\}$, then the stopping time τ^* is an optimal one and $\tau^* \leq \tau'$ $\mathbf{P}_z$-a.s. for any z and any optimal stopping time τ'.*

c) *The sequence $\tilde{V}_0(z) = g(z)$, $\tilde{V}_{k+1}(z) = \max[g(z), T\tilde{V}_k(z)]$ nondecreasingly converges to $V(z)$.*

The set D^* is called the stopping set and the set $C^* = X \setminus D^* = \{z : V(z) > g(z)\}$ is called the continuation set.

It is said often that statement c) offers a constructive method for finding the value function $V(z)$ (see, for example, [3], p. 19). Nevertheless, if $P_z[\tau^* > a] > 0$ for some $z \in X$ and any $a < \infty$ then $\tilde{V}_k(z) \leq \tilde{V}_{k+1}(z) < V(z)$ for all k. Let us consider the following example.

Example 1. Consider a random walk on the integer points of the interval [0, 32], where points 0 and 32 are absorbing ones, and on the others points the walk is symmetric Bernoulli one. Let $c(z) = 0$ and $g(z)$ is given by the following table:

z	0	1	2	3	4	5	6	7	8	9	10	11
$g(z)$	6	10	11	9	7.5	5	4.5	4.2	4	4.5	5.2	6

z	12	13	14	15	16	17	18	19	20	21
$g(z)$	6.9	7.3	7.7	8	7.7	7.3	6.8	6.2	5.7	5.3

z	22	23	24	25	26	27	28	29	30	31
$g(z)$	5	5.3	6	6.5	7.7	9	11	13	14	12

The function $g(z)$ is concave (i. e. $g(z) > (1/2)(g(z+1) + g(z-1))$) at points $1, 2, 3, 4, 12, 13, 14, 15, 16, 17, 18, 28, 29, 30, 31$ and convex at all others points (see Figure 1, where vertical intervals correspond to the values of the function $g(z)$ at corresponding points).

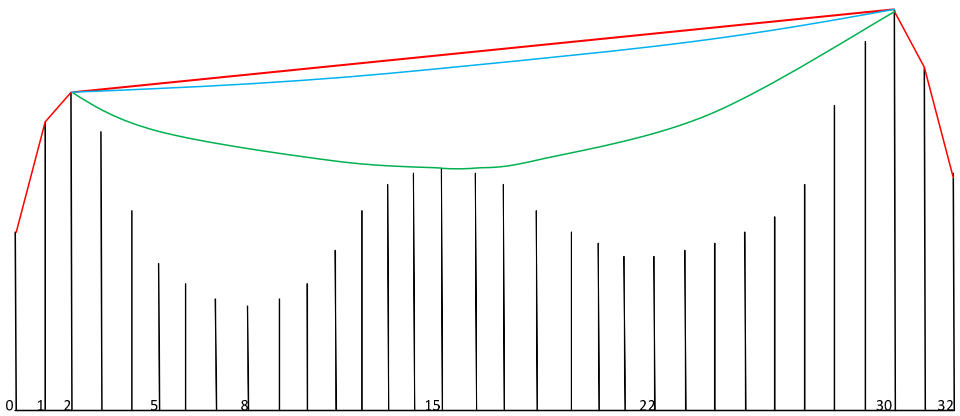

Figure 1: Recurrent calculation of $\tilde{V}_k(z)$

In this case $\tilde{V}_0(z) \equiv g(z)$, and $\tilde{V}_{k+1}(0) \equiv 6$, $\tilde{V}_{k+1}(32) \equiv 8$,

$$\tilde{V}_{k+1}(z) = \max\left[g(z),\ \frac{1}{2}\left(\tilde{V}_k(z-1) + \tilde{V}_k(z+1)\right)\right], \qquad 1 \le z \le 31, \qquad k \ge 0\,.$$

The value function $V(z)$ is depicted by red on Figure 1. The function $\tilde{V}_{53}(z)$ is depicted by green at the points, where it does not coincides with $g(z)$. The function $\tilde{V}_{350}(z)$ is depicted by blue. Here $\tilde{V}_{350}(16) = 11.77$, and $V(16) = 12.5$, so, even after 350 steps the approximation error is 5.84 percent.

If Z_n takes only finite number m of values then equation (3) can be solved by linear programming (see, for example, [2]). But under such approach the probabilistic meaning is lost and it is not clear how to generalize such approach even to the countable case. In [8, 9, 10] an algorithm to construct the value function $V(z)$ for the case of finite number of states was proposed. This algorithm is based on the elimination of the states and warranties that after no more than $(m-1)$ steps the set D^* will be found, and then after the same number of steps the value function $V(z)$ will be found for all z. It is mentioned in [11] that sometimes this algorithm allows to find $V(z)$ and D^* after finite number of steps also for countable case.

3. New approach to the solution of the problem. Arbitrary state space

For simplicity of exposition sometimes we shall assume that the following condition holds:

A. Functions $g(z)$ and $c(z)$ are bounded and there exist an absorbing state $e \in X$ and numbers $n_0 > 0$, $b < 1$, such that $\mathbf{P}_z\{Z_{n_0} = e\} \ge b > 0$ for any $z \in X$, and $g(e) = c(e) = 0$.

Remark 1. Condition **A** implies that the value function $V(z)$ is finite, $e \in D^*$ and therefore $\mathbf{P}_z[\tau^* < \infty] = 1$ for all $z \in X$. So Theorem 1 is applicable. A possibility to relax the condition **A** is discussed at the end of Section 3.

We consider sets $C \subset X$ and $D \subset X$ with or without indexes assuming that $D = X \backslash C$, and $C = X \setminus D$. Let I_C be an operator of multiplication by an indicator function of the set C, $I = I_X$.

Let τ_D, $0 \le \tau_D \le \infty$, be a random time when Z first time visits D. If $z \in D$, then $\tau_D = 0$. Denote

$$g_C(z) = E_z \left[\left(g(Z_{\tau_D}) - \sum_{k=0}^{\tau_D - 1} c(Z_k) \right) I_{\{\tau_D < \infty\}} \right]. \tag{4}$$

Main Lemma. a) *If* $z \in C$ *then* $Tg_C(z) = g_C(z)$.

b) *If* $C \subseteq \{z : Tg(z) \ge g(z)\}$ *and condition* **A** *is fulfilled then* $g(z) \le g_C(z) < \infty$ *for* $z \in C$ *and* $g_C(z) > g(z)$ *if* $z \in C$ *and* $Tg(z) > g(z)$.

Proof of the Main Lemma is given in Section 4.

Consider for the chain Z an optimal stopping problem with payoff function $g_C(z)$ and cost function $c(z)$.

Lemma 1. *Suppose that* $C \subseteq \{z : Tg(z) \ge g(z)\}$, $C \subseteq C^*$ *and condition* **A** *is fulfilled. Then the optimal stopping problem of the chain* Z *with payoff function* $g_C(z)$ *and cost function* $c(z)$ *has the same value function as the initial problem.*

Proof. Let $V_C^\tau(z) = E_z \left[g_C(Z_\tau) - \sum_{k=0}^{\tau-1} c(Z_k) \right]$, $V_C(z) = \sup_\tau V_C^\tau(z)$. According to the Main Lemma we have $g_C(z) \ge g(z)$, thus $V_C(z) \ge V(z)$. On the other side for any τ there exists τ_1, such that $V^{\tau_1}(z) = V_C^\tau(z)$. Indeed, for those ω where $Z_\tau \in D$, we set $\tau_1 = \tau$, and for those ω where $Z_\tau \in C$, we take as τ_1 the time of the first after τ visit the set D. It completes the proof of Lemma 1.

Let us define now a sequence of sets C_k and functions $g_k(z)$, $k \ge 0$, as follows: $C_0 = \emptyset$, $g_0(z) = g(z)$, and if C_l, $g_l(z)$ are defined for $0 \le l \le k$, $k \ge 0$, then

$$C_{k+1} = C_k \bigcup \{z : g_k(z) < Tg_k(z)\}, \qquad g_{k+1}(z) = g_{k,C_{k+1}}(z), \tag{5}$$

where $g_{k,C_{k+1}(z)}$ is constructed from $g_k(z)$ using formula (4) as the expected reward at the time of the first visit to D_{k+1}. Note that by the strong Markov property and monotonicity of sequence C_k, $k \ge 0$, the function $g_{k+1}(z)$ can be constructed using $g(z)$ instead of $g_k(z)$, so that $g_{k+1}(z) = g_{C_{k+1}}(z)$. Note also that if there exists k_0, such that $\{z : g_{k_0}(z) < Tg_{k_0}(z)\} = \emptyset$, then $g_k(z) = g_{k_0}(z)$, $C_k = C_{k_0}$ for $k \ge k_0$.

Now we can prove the main theorem.

Theorem 2. *If condition* **A** *is fulfilled then the sequence* C_k, $k \ge 0$, *does not decrease and tends to the continuation set* C^* *in the problem of optimal stopping of the Markov chain* Z *with payoff function* $g(z)$ *and cost function* $c(z)$, *and the sequence* $g_k(z)$, $k \ge 0$, *does not decrease and tends to the corresponding value function* $V(z)$.

Proof. Using Main Lemma we obtain that the sequence $g_k(z)$, $k \ge 0$, does not decrease. Since it is bounded from above by the finite function $V(z)$, we obtain that the sequence $g_k(z)$, $k \ge 0$, has a limit which we denote by $\tilde{V}(z)$. By the definition the sequence C_k, $k \ge 0$, does not decrease and therefore has a limit which we denote by $\tilde{C}$.

For any $z \in \tilde{C}$ there exists $k(z)$, such that $z \in C_k$ for $k \ge k(z)$, and therefore by the Main Lemma $Tg_k(z) = g_k(z) > g(z)$ for $k > k(z)$. Hence

$$T\tilde{V}(z) = \tilde{V}(z) > g(z) \quad \text{for } z \in \tilde{C}. \tag{6}$$

If $z \in \tilde{D}$, then $z \in D_k$ for any $k \geq 0$, and therefore $Tg_k(z) \leq g_k(z) = g(z)$. Hence

$$T\tilde{V}(z) \leq g(z) = \tilde{V}(z) \text{ for } z \in \tilde{D}. \tag{7}$$

Thus $\tilde{V}(z)$ satisfies (3) and inequality $\tilde{V}(z) \leq V(z)$. According to statement a) of Theorem 1 the value function $V(z)$ is a minimal solution of (3), consequently $\tilde{V}(z) = V(z)$, $\tilde{C} = C^*$. This completes the proof of Theorem 2.

Example 2. Consider the same optimal stopping problem as in the example 1. On Figure 2 the function $g_1(z)$ is depicted by yellow at the points, where it does not coincide with $g(z)$, the function $g_2(z)$ is depicted by green, the function $g_3(z)$ is depicted by blue, the function $g_4(z)$ is depicted by black, the function $g_5(z)$ is depicted by red. So $g_5(z) = V(z)$, and after five steps we found the value function $V(z)$ and the stopping set $D^* = D_5$. Here $D_5 = \{0, 1, 2, 30, 31, 32\}$; $D_4 = D_5 \bigcup \{15\}$; $D_3 = D_4 \bigcup \{14, 16\}$; $D_2 = D_3 \bigcup \{3, 13, 17, 29\}$; $D_1 = D_2 \bigcup \{4, 12, 18, 28\}$.

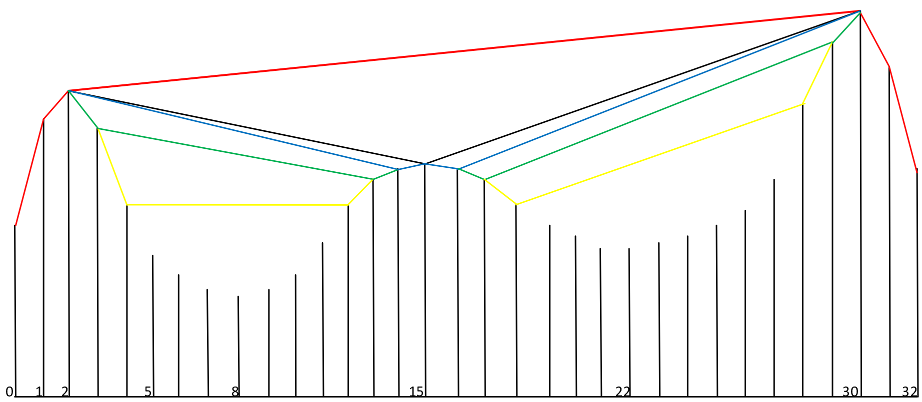

Figure 2: Proposed procedure of finding $V(z)$

Remark 2. Let X consists of $m < \infty$ states. As a rule $P_z[\tau^* > a] > 0$ for any $a < \infty$ at least for some $z \in X$ and one needs infinite number of steps to obtain the value function using the constructive method. The proposed procedure warranties that the value function will be found not more than for $(m-1)$ steps.

Remark 3. The statement of Theorem 2 is valid in essentially more general situation than under condition **A**. It seems that if the value function is finite and the probability to reach the stopping set is one for each point of X, then the result is true. The author plans to investigate this question in a future work.

4. Proof of the Main Lemma

For any $C \subset X$ we define an operator $\mathcal{P}_C$ as follows:

$$\mathcal{P}_C f := \mathcal{P} I_C f,$$

where I_C is an operator of multiplication by the indicator function of C.

Let $e \in D$ and $\tau'_D,\ 1 \le \tau'_D \le \infty$, be the first (after zero) passage time of the set D for Z. If $z \in C$, then $\tau'_D = \tau_D$, and if $z \in D$, then it is the time of the first return. Consider

$$H_C g(z) = E_z\left[\left(g(Z_{\tau'_D}) - \sum_{k=0}^{\tau'_D - 1} c(Z_k)\right) I_{\{\tau'_D < \infty\}}\right]. \tag{8}$$

Note that

$$H_C g(z) = g_C(z) \text{ for } z \in C. \tag{9}$$

Lemma 2. *If condition* **A** *is fulfilled then the following representations hold*:

$$H_C g(z) = \sum_{l=0}^{\infty} \mathcal{P}_C^l \left(\mathcal{P}_D g(z) - c(z)\right) = \left(I - \mathcal{P}_C\right)^{-1}\left(\mathcal{P}_D g(z) - c(z)\right), \tag{10}$$

$$H_C g(z) = T g(z) + \left(I - \mathcal{P}_C\right)^{-1} \mathcal{P}_C (T g(z) - g(z)), \tag{11}$$

$$H_C g(z) = \mathcal{P}_D g(z) - c(z) + \mathcal{P}_C H_C g(z). \tag{12}$$

Proof. The second equality in (10) follows from the definition of the operator $\left(I - \mathcal{P}_C\right)^{-1}$. The convergence of the series and the existence of $\left(I - \mathcal{P}_C\right)^{-1}$ follows from the condition **A**. To prove the first equality let us show that

$$E_z\left[g(Z_{\tau'_D}) I_{\{\tau'_D < \infty\}}\right] = \sum_{l=0}^{\infty} \mathcal{P}_C^l \mathcal{P}_D g(z), \quad E_z\left[\sum_{k=0}^{\tau'_D - 1} c(Z_k) I_{\{\tau'_D < \infty\}}\right] = \sum_{l=0}^{\infty} \mathcal{P}_C^l c(z). \tag{13}$$

The first equality in (13) corresponds to the total probability formula with respect to the partition $\{\tau'_D = l+1\}$, $l \ge 0$. Indeed, $\left(\mathcal{P}_C\right)^l \mathcal{P}_D g(z)$ is a result of averaging of $g(Z_{l+1})$ over trajectories that start at z, then l moments spend at C, and after that enter D. The proof of the second equality in (13) is analogous. The first equality in (10) follows from (13).

Substituting the equality $\mathcal{P}_D = \mathcal{P} - \mathcal{P}_C$ into the left-hand side of (10) and after that using equality $\left(I - \mathcal{P}_C\right)^{-1} = I + \left(I - \mathcal{P}_C\right)^{-1} \mathcal{P}_C$ we obtain

$$\begin{aligned} H_C g(z) &= \left(I - \mathcal{P}_C\right)^{-1}\left(\mathcal{P} g(z) - c(z) - \mathcal{P}_C g(z)\right) \\ &= \left(I + \left(I - \mathcal{P}_C\right)^{-1} \mathcal{P}_C\right) T g(z) - \left(I - \mathcal{P}_C\right)^{-1} \mathcal{P}_C g(z) \\ &= T g(z) + \left(I - \mathcal{P}_C\right)^{-1} \mathcal{P}_C T g(z) - \left(I - \mathcal{P}_C\right)^{-1} \mathcal{P}_C g(z)\ . \end{aligned} \tag{14}$$

This is equivalent to (11). It follows from the first equality in (10) that

$$H_C g(z) = \mathcal{P}_D g(z) - c(z) + \sum_{l=1}^{\infty} \mathcal{P}_C^l \left(\mathcal{P}_D g(z) - c(z)\right) = \mathcal{P}_D g(z) - c(z) + \mathcal{P}_C H_C g(z). \tag{15}$$

It completes the proof of Lemma 2.

Note that formula (11) for $z \in D$ was obtained in [11] for countable case. A version of (11) was obtained in [4] and [5].

Now we can prove the Main Lemma.

Proof of the Main Lemma. Using equality $Tf(z) = \mathcal{P}_C f(z) + \mathcal{P}_D f(z) - c(z)$, equality (12) and (9) we get

$$\begin{aligned} I_C T g_C(z) &= I_C\left(\mathcal{P}_C g_C(z) + \mathcal{P}_D g_C(z) - c(z)\right) \\ &= I_C\left(\mathcal{P}_C H_C g(z) + \mathcal{P}_D g(z) - c(z)\right) = I_C H_C g(z) = I_C g_C(z). \end{aligned} \tag{16}$$

Since $\mathcal{P}_C(Tg(z) - g(z)) \geq 0$ under conditions of Main Lemma, it follows from (11) and (9) that $g_C(z) \geq Tg(z)$ for $z \in C$. The proof for the strict inequality is the same. It completes the proof of the Main Lemma.

5. Possibilities for a generalization to the continuous time

In this section we shall consider the case of continuous time. The general theory of the optimal stopping and methods of constructing the value function can be found, for example, in [3], [6], [1]. The goal of this section is to demonstrate by some examples how the proposed procedure can be generalized to the case of one-dimensional diffusion ξ_t with functional $E_z[g(\xi_\tau)]$. The idea is the same as in the discrete time.

For any open interval C denote by $g_C(z)$ the expected reward at the time of the first visit the complement of C. Then $g_C(z) = g(z)$ for $z \in D = X \setminus C$ and $Lg_C(z) = 0$ for $z \in C$, where L is a differential operator corresponding to the diffusion (see, for example, [3] Sections 4.5, 7.1). So, the operator L plays in the continuous time the role of the operator $T - I$ in the discrete time. Suppose we found C such that $g_C(z) > g(z)$ on C. Then $C \in C^*$, where C^* is the continuation set, and the problem of optimal stopping with the payoff function $g_C(z)$ has the same value function as the initial one. The proof is the same as the proof of Lemma 1. Indeed, the value function corresponding to $g_C(z)$ is greater than or equal to the value function corresponding to $g(z)$ since $g_C(z) \geq g(z)$. On the other hand for each τ we can define

$$\tau' := \begin{cases} \tau & \text{if } \xi_\tau \in D\,, \\ \inf[s : s > 0, \xi_{\tau+s} \in D] & \text{if } \xi_\tau \in C\,. \end{cases} \tag{17}$$

Then $g(\xi_{\tau'}) = g_C(\xi_\tau)$ and hence the value functions coincide.

For the new payoff function we can try similarly to find intervals which certainly belong to C^*. Repeating this procedure we obtain finally a set $\tilde{C}$ and the modified payoff function $g_{\tilde{C}}(z)$ such that there is no point in $\tilde{D} = X \setminus \tilde{C}$ such that in the neighborhood of this point we can increase the reward. In such situation $\tilde{C} = C^*$ and $g_{\tilde{C}}(z) = V(z)$. In our examples intervals which belong to C^* for sure are:

a) one-side neighborhoods of points of discontinuity of $g(z)$;
b) intervals where $Lg(z) > 0$;
c) neighborhoods of points, where $g'_-(a) < g'_+(a)$;
d) neighborhoods of points of singularities of the diffusion.

Example 3. We consider a standard Wiener process w_t with initial point in $(-1, 1)$, stopped at the points -1 and 1, with the functional $E_z\left[g(w_\tau)\right]$. We suppose that the set of discontinuity of functions $g(z)$, $g'(z)$, $g''(z)$ is finite, the set of isolated zeros of the function $g''(z)$ is also finite, the function $g(z)$ is upper semi-continuous, i. e. $g(z) \geq \limsup_{x\to z} g(x),\ z \in [-1, 1]$.

Recall (see, for example, [3] p. 145) that the differential operator corresponding to this process is $Lf(z) = (1/2)f''(z)$. So, for any interval (a,b) the expected reward at the time of the first exit from (a,b) is equal to

$$g_{(a,b)}(z) := g(a) + \frac{g(b)-g(a)}{b-a}(z-a) \quad \text{for} \quad a \le z \le b\,.$$

The solution of the problem is well known (see, for example, [3] p. 146): the value function coincides with the minimal convex majorant of the payoff function. We propose the following procedure for constructing the value function.

1) At the first stage we change the payoff function in a neighborhood of each point of discontinuity $g(z)$. We change it in such a way, that the new payoff function is continuous and the problem of optimal stopping with the new payoff function has the same value function as the initial problem.

Let $g(a) > \lim_{z\downarrow a} g(z)$ for some $a \in (-1,1)$. Due to our assumptions about function $g(z)$ this limit exists. We can choose $\varepsilon > 0$ such that there exist no points of change of sign of $g''(z)$, no points of discontinuity on the interval $(a, a+\varepsilon)$, and $g_{(a,a+\varepsilon)}(z) > g(z),\ z \in (a, a+\varepsilon)$ (see Figure 3 (i)).

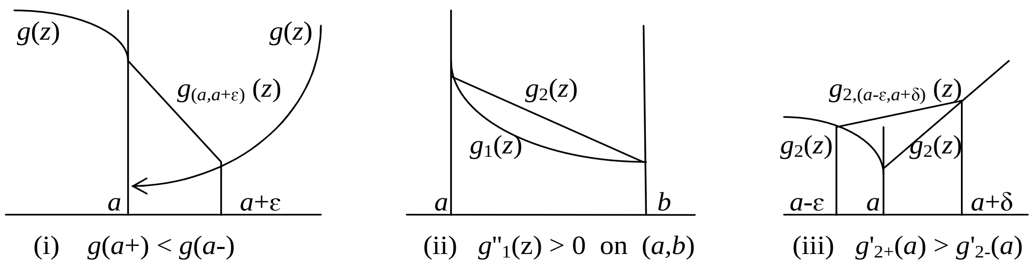

Figure 3: Example 3

Therefore the problem of optimal stopping with the payoff function $g_{(a,a+\varepsilon)}(z)$ has the same value function as the initial problem. The same situation holds for the points where $g(a) > \lim_{z\uparrow a} g(z)$. Now we consider function $g_1(z)$ which is obtained from $g(z)$ using the mentioned procedure for all points of discontinuity of $g(z)$ and let $C_1 = \{z :\ g_1(z) > g(z)\}$. Note that the function $g_1(z)$ is continuous on $[-1;1]$, functions $g_1'(z)$, $g_1''(z)$ have only finite number of points of discontinuity, and the problem of optimal stopping with the payoff function $g_1(z)$ has the same value function as the initial problem.

2) At the second stage we change $g_1(z)$ on intervals, where $g_1''(z) > 0$. Let $C_2 = C_1 \bigcup \{z :\ g_1''(z) > 0\}$. Due to our assumptions about function $g(z)$ the set C_2 consists of the finite number of open intervals. Denote by A the set of such intervals. Let $g_2(z) = g_1(z)$ for $z \notin C_2$ and $g_2(z) = g_{1,(a,b)}(z)$ for $a \le z \le b$ and any $(a,b) \in A$, where, as earlier, $g_{1,(a,b)}(z)$ is the expected reward at the time of the first exit from (a,b) for the payoff function $g_1(z)$. Then $g_2(z) \ge g_1(z)$ (see Figure 3 (ii)), $C_2 \subseteq C^*$ and the problem with the functional $E_z\,[g_2(w_\tau)]$ has the same value function as the initial problem. Note that $g_2''(z) \le 0$ for all points of continuity, the function $g_2(z)$ is continuous and the functions $g_2'(z)$, $g_2''(z)$ have only finite number of points of discontinuity.

3) Consider now the points of discontinuity of $g_2'(z)$. Denote by $g_{2+}'(z)$ the right and by $g_{2-}'(z)$ the left derivative of $g_2(z)$. The existence of these derivatives follows from our assumptions about function $g(z)$. If $g_{2-}'(a) < g_{2+}'(a)$, $-1 < a < 1$, then there exist $\varepsilon > 0$ and $\delta > 0$ such that $g_{2-}'(a-\varepsilon) < g_{2,(a-\varepsilon,\,a+\delta)}'(z) < g_{2+}'(a+\delta)$, $g_{2,\,(a-\varepsilon,\,a+\delta)}(z) > g_2(z)$ for $z \in (a-\varepsilon, a+\delta)$ (see Figure 3 (iii)). This follows from the fact, that we can choose ε

and δ in such a way, that there are no points of change of sign of $g_2''(z)$ and no points of discontinuity on the interval $(a-\varepsilon, a+\delta)$.

Set $g'_{2-}(-1) = +\infty$, $g'_{2+}(1) = -\infty$. Since $g_2''(z) \le 0$ for all points of continuity, by increasing ε and δ we obtain that there exist minimal values of ε and δ – denote them by ε_1 and δ_1 – such that $g'_{2-}(a-\varepsilon_1) \ge g'_{2,(a-\varepsilon_1,a+\delta_1)}(z) \ge g'_{2+}(a+\delta_1)$ for $z \in (a-\varepsilon_1, a+\delta_1)$. This is an analog of the smooth fitting condition in the case of smooth $g(z)$. Note that for each fixed z the function $g'_{2,(a-\varepsilon,a+\delta)}(z)$, as a function on ε and δ increases on ε and δ for $\varepsilon < \varepsilon_1$, $\delta < \delta_1$.

Set $C_3 = C_2 \bigcup (a-\varepsilon_1, a+\delta_1)$ and $g_3(z) = g_{2,(a-\varepsilon_1,a+\delta_1)}(z)$ for $z \in (a-\varepsilon_1, a+\delta_1)$. It is obvious that $g_3(z) \ge g_2(z)$, $C_3 \subseteq C^*$, the problem with the functional $E_z\,[g_3(w_\tau)]$ has the same value function as the initial problem, and the number of points of discontinuity of $g_3'(z)$ such that $g'_{3-}(z) < g'_{3+}(z)$ is less then the number of such points for $g_2'(z)$. We can apply to $g_3(z)$ the same procedure as we applied to $g_2(z)$. Since the number of points where $g'_{2-}(z) < g'_{2+}(z)$ is finite, after finite number of steps we obtain a set $\tilde{C}$, and a function $\tilde{g}(z)$ such that:

a) $\tilde{g}(z) \ge g(z)$, $\quad \tilde{g}(z) = E_z\,[g(w_{\tilde{\tau}})]$, where $\tilde{\tau} = \inf\{t \ge 0 : \ w_t \notin \tilde{C}\}$,
b) $\tilde{g}''(z) = 0$ for $z \in \tilde{C}$, $\tilde{g}'_-(z) \ge \tilde{g}'_+(z)$ for all $z \in (-1,1)$, and $\tilde{g}''(z) \le 0$ for all points of continuity.

It follows from a) that the value function is the same in the problem of the optimal stopping with the payoff function $g(z)$ and with the payoff function $\tilde{g}(z)$. It follows from b) that $\tilde{g}(z)$ is convex and coincides with its minimal convex majorant. Consequently $\tilde{g}(z) = V(z)$ and $\tilde{C} = C^*$.

Remark 4. One can say that an interval (a,b) in the problem of Example 3 with a smooth $g(z)$ is a smooth fitting interval if the function $g_{(a,b)}(z)$ has the same derivative as $g(z)$ at points a and b. Any smooth fitting interval gives a solution of the Stefan free-boundary problem. It can happen that such interval has no relation to the set C^* and for checking that the solution of the Stefan free-boundary problem coincides with the value function one uses usually a verification theorem. In the proposed procedure we do not need to use a verification theorem. We think, that in essentially more general situation instead of a verification theorem it suffices to prove that if the *payoff function* satisfies the conditions: $g'_-(z) \ge g'_+(z)$ for all z, and $Lg(z) \le 0$ for all points of continuity, then $g(z) = V(z)$.

Example 4. We consider a Wiener process $w_{1,t}$ on the interval $[-1;1]$ with the absorbtion at the points -1 and 1, a partial reflection to the right with probability α, $0 < \alpha < 1$, at the point 0, and the functional $E_z\,[g(w_{1,\tau})]$ where $g(z)$ is the same function as in Example 3.

The differential operator corresponding to this process is $L_1 f(z) = (1/2)f''(z)$ for $z \neq 0$ with the condition $(1+\alpha)f'_+(0) - (1-\alpha)f'_-(0) = 0$.

At first we use the same procedure as in Example 3 for interval $[-1,0]$ assuming that points $z=-1$ and $z=0$ are absorbing. Then we use the same procedure for interval $[0,1]$. As a result we obtain the continuous function $g_1(z)$ and the set C_1 such that $C_1 \subseteq C^*$ and the problem of optimal stopping with functional $E_z\,[g_1(w_{1,\tau})]$ has the same value function as the initial problem. The set C_1 consists of the final number of open intervals, the functions $g_1'(z)$ and $g_1''(z)$ have only finite number of points of discontinuity, $g_1''(z) = 0$ for $z \in C_1$, $g_1''(z) \le 0$ for all points of continuity, $g'_{1-}(z) \ge g'_{1+}(z)$ for all $z \in (-1,0)$ and $z \in (0,1)$. So, $g_1(z)$ is concave for $z \in (-1,0)$ and $z \in (0,1)$.

Let us consider the point $z=0$. For any a, b, $-1 \le a < b \le 1$, denote by $g_{(a,b)}(z)$

the expected reward at the time of the first exit from (a, b) for payoff function $g_1(z)$. If either $b \le 0$ or $a \ge 0$ then due to our construction $g_{(a,b)}(z) \le g_1(z)$ for all $z \in [-1, 1]$. If $a < 0$, $b > 0$, then the function $g_{(a,b)}(z)$ satisfies to the conditions: $L_1 g_{(a,b)}(z) = 0$ for $z \in (a, 0) \bigcup (0, b)$, $(1+\alpha)g'_{(a,b)}(+0) - (1-\alpha)g'_{(a,b)}(-0) = 0$, $g_{(a,b)}(a) = g_1(a)$, $g_{(a,b)}(b) = g_1(b)$.

Therefore, if $a < 0$, $b > 0$, then $g_{(a,b)}(0) = \dfrac{b(1-\alpha)g_1(a) - a(1+\alpha)g_1(b)}{b(1-\alpha) - a(1+\alpha)}$ and

$$g_{(a,b)}(z) = \begin{cases} [g_{(a,b)}(0)(z-a) - g_1(a)z]/(-a) & \text{for } z \in (a, 0), \\ [g_1(b)z + g_{(a,b)}(0)(b-z)]/b & \text{for } z \in (0, b). \end{cases} \tag{18}$$

If $(1+\alpha)g'_{1+}(0) - (1-\alpha)g'_{1-}(0) \le 0$ then from the mentioned concavity of $g_1(z)$ and (18) follows that $g_{(a,b)}(z) \le g_1(z)$ for all $-1 \le a,\ b,\ z \le 1$, and consequently $g_1(z) = V(z)$ and $C_1 = C^*$.

If $(1+\alpha)g'_{1+}(0) - (1-\alpha)g'_{1-}(0) > 0$ then there exist $\varepsilon > 0$ and $\delta > 0$ such that $g'_{1-}(-\varepsilon) < g'_{+(-\varepsilon,\,\delta)}(-\varepsilon)$, $g'_{-(-\varepsilon,\,\delta)}(\delta) < g'_{1+}(\delta)$, $g_{(-\varepsilon,\delta)}(z) > g_1(z)$ for $z \in (-\varepsilon, \delta)$ (see Figure 4 (a)).

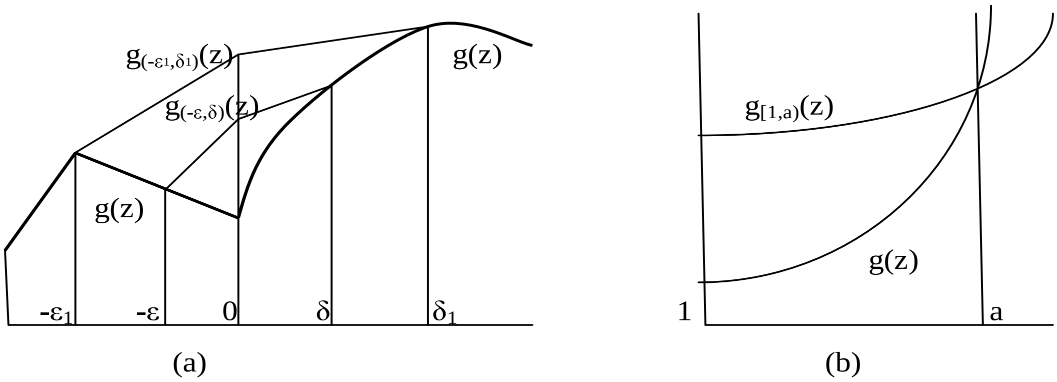

Figure 4: Examples 4 and 5

Since $g_1(z)$ is concave for $z \in (-1, 0)$ and $z \in (0, 1)$, increasing ε and δ we obtain that there exist minimal values of ε and δ – denote them by ε_1 and δ_1 – such that $g'_{1-}(\delta_1) \ge g'_{(-\varepsilon_1,\delta_1)}(-0)$, $g'_{(-\varepsilon_1,\delta_1)}(+0) \ge g'_{1+}(\delta_1)$. Note that for each fixed z the function $g_{(-\varepsilon,\delta)}(z)$, as a function on ε and δ increases on ε and δ for $\varepsilon < \varepsilon_1$, $\delta < \delta_1$. As a result we have $C^* = C_1 \bigcup (-\varepsilon_1, \delta_1)$, $V(z) = g_1(z)$ for $z \notin (-\varepsilon_1, \delta_1)$, $V(z) = g_{(-\varepsilon_1,\delta_1)}(z)$ for $z \in (-\varepsilon_1, \delta_1)$.

Example 5. Geometric Brownian motion x_t on $[1; \infty]$ with parameters $(-r, \sigma)$, a killing intensity λ, a reflection at the point 1 and with the functional $E_z\,[g(x_\tau)]$. We assume that the function $g(z)$ satisfies to the same conditions of continuity and differentiability as in Example 3 and the set of the isolated zeros of the function $L_2 g(z) := \dfrac{\sigma^2 z^2}{2} g''(z) - rzg'(z) - \lambda g(z)$ is finite. Let $\kappa_+ > 1$ and $\kappa_- < 0$ be the solutions of the equation $\kappa^2 - \left(1 + \dfrac{2r}{\sigma^2}\right)\kappa - \dfrac{2\lambda}{\sigma^2} = 0$. We assume also that: a) $\lim_{z\to\infty} |g(z)| z^{-\kappa_+} < \infty$, b) $L_2 g(z) < 0$ for $z \ge z_1 \ge 1$, and c) $g'(1) > 0$.

It is well known (see, for example, [3] formula (26.1.18)) that the differential operator corresponding to this process is $L_2 f(z)$ with boundary condition $f'(1) = 0$. First of all we investigate the behavior of $g(z)$ at point 1. Denote by $g_{[1,a)}(z)$ the expected reward at the time of the first exit from $[1, a)$. Then $g_{[1,a)}(z) = g(z)$ for $z \ge a$ and for $z \in (1, a)$

it satisfies to the equation $L_2 g_{[1,a)}(z) = 0$, with boundary conditions $g_{[1,a)}(a) = g(a)$, $g'_{[1,a)}(1) = 0$. Therefore

$$g_{[1,a)}(z) := \frac{g(a)\,(\kappa_+ z^{\kappa_-} - \kappa_- z^{\kappa_+})}{\kappa_+ a^{\kappa_-} - \kappa_- a^{\kappa_+}} \text{ for } z \in [1,a)\,. \tag{19}$$

It follows from c) and the conditions on function $g(z)$ that if $a-1$ is small enough then $g'_{[1,a)}(a) < g'(a)$ and $g_{[1,a)}(z) > g(z)$ for $z \in [1,a)$ (see Figure 4 (b)).

Thus $[1,a) \subseteq C^*$ and the problem with the functional $E_z\left[g_{[1,a)}(w_\tau)\right]$ has the same value function as the initial problem. Now we shall use the same procedure as in Example 3, but we shall change $g_{[1,a)}(z)$ on each interval (b,c) from C^* to a function $f(z) = B_1 z^{\kappa_-} + B_2 z^{\kappa_+}$, where B_1 and B_2 are chosen from the condition $f(b) = g_{[1,a)}(b)$, $f(c) = g_{[1,a)}(c)$ in case $b > 1$ and $f'(b) = 0$, $f(c) = g_1(c)$ in case $b = 1$, which coincides with the expected reward at the time of the first exit from (b,c). After finite number of steps we obtain the stopping set and the value function. It is simple to check that from conditions a) and b) follows that the value function is finite and the set C^* is bounded. Note that the case $g(z) = z$ corresponds to the Russian option (see [3], Section 26). Since in this case $L_2 g(z) = -(r+\lambda)z < 0$, the only point where we can locally increase payoff function without changing the value function is the point $z = 1$. and one has only one step. The optimal value a^* in (19) can be found as earlier from the condition $a^* = \{\inf a : g'_{[1,a)}(a) \geq g'(a) \equiv 1\}$.

Example 6. We consider a standard Wiener process w_t with an initial point $z \in (-\infty, +\infty)$ and a functional $E_z\left[e^{-\lambda\tau} g(w_\tau)\right]$. Such problem is equivalent to the problem with functional $E_z\left[g(\tilde{w}_\tau)\right]$, where $\tilde{w}_t$ is a standard Wiener process with a killing intensity λ. The differential operator corresponding to this process is

$$L_3 f(z) = (1/2) f''(z) - \lambda f(z)\,. \tag{20}$$

For the simplicity we suppose that $g(0) = 0$, $L_3 g(z) < 0$ for $z \neq 0$, $g'_+(0) = b > 0 > g'_-(0) = a$. The payoff function $g(z) = az$ for $z \leq 0$ and $g(z) = bz$ for $z \geq 0$ satisfies to this conditions.

Since $L_3 g(z) < 0$ for all $z \neq 0$, and $a = g'_-(0) < g'_+(0) = b$, the only point where we can locally increase payoff function without changing the value function is the point $z = 0$.

Let $\tau(c,d)$ be the time of the first visit of the complement of the interval (c,d), where $-\infty < c < d < +\infty$. As earlier we define the expected reward at the time $\tau(c,d)$ as

$$g_{(c,d)}(z) = E_z[g(w_{\tau(c,d)})]. \tag{21}$$

Then

$$L_3 g_{(c,d)}(z) \equiv 0 \text{ for } z \in (c,d), \quad g_{(c,d)}(c) = g(c), \quad g_{(c,d)}(d) = g(d)\,. \tag{22}$$

Let $\mathbf{B}$ be the set of intervals such that $c < 0 < d$ and

$$a = g'_-(c) \leq g'_{+,(c,d)}(c), \quad g'_{-,(c,d)}(d) \leq g'_+(d) = b. \tag{23}$$

We shall use the following properties of $\mathbf{B}$ which are valid in essentially more general situation. They follow from the fact that $L_3 g(z) \leq 0$ for all points of continuity of $g''(z)$. The proof of this properties is analogous to the proof of step 3 in the Example 1.

1) If $(c,d) \in \mathbf{B}$ then $g_{(c,d)}(z) > g(z)$, $(c,d) \subseteq C^*$ and the problem with the payoff function $g_{(c,d)}(z)$ has the same value function as the problem with the payoff function $g(z)$.

2) If $c < 0 < d$ and $c, |d|$ are small enough then $(c,d) \in \mathbf{B}$ and both inequalities in (23) are strong.

3) If $(c,d) \in \mathbf{B}$ and the first (or the second) inequality in (23) is strong, then there exists $c_1 < c$ (or $d_1 > d$) such that $(c_1, d) \in \mathbf{B}$ (or $(c, d_1) \in \mathbf{B}$) and $g_{(c_1,d)}(z) > g_{(c,d)}(z)$ for $z \in (c_1, d)$ (or $g_{(c,d_1)}(z) > g_{(c,d)}(z)$ for $z \in (c, d_1)$).

Let (c^*, d^*) be the minimal interval for which $c^* < 0 < d^*$ and $g'_-(c^*) \geq g'_{+,(c^*,d^*)}(c^*)$, $g'_{-,(c^*,d^*)}(d^*) \geq g'_+(d^*)$.

4) If $|c^*|, d^* < \infty$ then $(c^*, d^*) \in \mathbf{B}$ and $g'_-(c^*) = g'_{+,(c^*,d^*)}(c^*)$, $g'_{-,(c^*,d)^*}(d^*) = g'_+(d^*)$.

In case $|c^*|, d^* < \infty$ the function $g(c^*, d^*)(z)$ is smooth and $L_3 g(c^*, d^*)(z) \leq 0$ for all $z \neq c^*, d^*$. Using a standard methods one can show that the value function in the problem of optimal stopping with payoff function $g_{(c^*,d^*)}(z)$ coincides with $g_{(c^*,d^*)}(z)$. It follows from here that in the initial problem the value function coincides with $g_{(c^*,d^*)}(z)$, $(c^*, d^*) = C^*$ and $\tau(c^*, d^*)$ is the optimal stopping time. So, we need just to construct the values c^*, d^*.

Let us consider the case $g(z) = az$ for $z \leq 0$ and $g(z) = bz$ for $z \geq 0$. Without restriction of generality we suppose that $\lambda = 1/2$. Consider the function

$$\psi(z, c, d) = bd\frac{\sinh(z-c)}{\sinh(d-c)} + ac\frac{\sinh(d-z)}{\sinh(d-c)}\,. \tag{24}$$

This function satisfies equation (22) and consequently

$$\psi(z, c, d) = g_{(c,d)}(z) \text{ for } z \in (c, d). \tag{25}$$

The values c^*, d^* are the roots of the system of equations $\psi'_z(c, c, d) = a$, $\psi'_z(d, c, d) = b$, which can be written in the form

$$bd - ac\coth(d-c) = a\sinh(d-c)\,, \tag{26}$$

$$bd\coth(d-c) - ac = b\sinh(d-c)\,. \tag{27}$$

If $a = -b$ then $c^* = -d^*$ and (26) follows from (27). From (27) and equalities $a = -b$, $c^* = -d^*$ we get $bd^*(\coth(2d^*) - 1) = b\sinh(2d^*)$. It is simple to show that this equation has a unique root d^* which is the same for all values of b.

Let $a \neq -b$. The system (26)–(27) can be rewritten as

$$b^2 d + a^2 c = ab(c+d)\coth(d-c)\,, \tag{28}$$

$$b^2 d^2 - a^2 c^2 = ab(c+d)\sinh(d-c)\,, \tag{29}$$

or using the equality $\coth^2(x) - \sinh^2(x) = 1$ as

$$(b^2 d + a^2 c)^2 - (b^2 d^2 - a^2 c^2)^2 = a^2 b^2 (c+d)^2\,, \tag{30}$$

$$b^2 d^2 - a^2 c^2 = ab(c+d)\sinh(d-c)\,. \tag{31}$$

Equation (30) can be represented in the form

$$(b^2 d^2 - a^2 c^2)[(b^2 d^2 - a^2 c^2) - (b^2 - a^2)] = 0\,. \tag{32}$$

Solution $dc = ab$ of (32) contradicts to (31). So, optimal values c^*, d^* are the roots of the system

$$b^2 d^2 - a^2 c^2 = b^2 - a^2\,, \tag{33}$$

$$b^2 - a^2 = ab(c+d)\sinh(d-c)\,. \tag{34}$$

Solving (33) with respect to c and substituting the result into (34) we obtain the equation with respect to d^* which has a unique positive solution.

Remark 5. We believe that the proposed procedure can be generalized to a much more general situation and also to a multi-dimensional case.

Acknowledgement

The author would like to thank V.I. Arkin, A.D. Slastnikov for useful discussions, and I.M. Sonin, Yu.M. Kabanov, two referees, and AE for very valuable remarks and suggestions.

This work was partly supported by RFBR grant 10-01-00767-a

References

[1] S. Dayanik, I. Karatzas, *On the optimal stopping problem for one-dimensional diffusions.* Stochastic Processes & Applications, 107, 2003, pp. 173-212.

[2] R. Feldman, C.Valdez-Flores, *Applied Probability and Stochastic Processes*, PWS Publ. Co, 1995.

[3] G. Peskir, A.N. Shiryaev, *Optimal Stopping and Free-Boundary Problems*, Birkhauser, Basel, 2006.

[4] E. L. Presman, *On Sonin's algorithm for solution of the optimal stopping problem*, Proceedings of the Fourth international conference on control problems (January 26—30, 2009). — M.: Institute of Control Sciences, 2009, pp. 300–309.

[5] E. L. Presman , I. M. Sonin, *On one problem of optimal stopping for random variables given on a Markov chain*, Theory of Probab. and its Applic., v. 54 (3), 2009.

[6] P. Salminen, *Optimal stopping of one-dimensional diffusions*, Mathematische Nachrichten, 124, 2085, pp. 85-101.

[7] A.N. Shiryayev, *Statistical Sequential Analysis: Optimal Stopping Rules*, Nauka, Moscow, 1969 (in Russian). English translation of the second edition: A.N. Shiryayev, *Optimal Stopping Rules*, Springer-Verlag, NY, 1978.

[8] I.M. Sonin, *Two simple theorems in the problems of optimal stopping*, In: Proc. 8th INFORMS Appl. Prob. Conf., Atlanta, Georgia, 1995, p. 27.

[9] I.M. Sonin, *The elimination algorithm for the problem of optimal stopping*, Math. Meth. of Oper. Res., 49 (1999), pp. 111-123.

[10] I.M. Sonin, *The state reduction and related algorithms and their applications to the study of Markov chains, graph theory and the optimal stopping problem*, Advances in Math., 145 (1999), pp. 159-188.

[11] I.M. Sonin, *Optimal stopping of Markov chains and recursive solution of Poisson and Bellman equations*, In: From Stochastic Calculus to Mathematical Finance. The Shiryaev Festschrift, Springer-Verlag, 2006, pp. 609-621.