.

About new approach to the solution of the optimal stopping problem

Ernst Presman

Central Economics & Mathematics Institute of Russian Academy of Science

.

.

International Conference "Stochastic Optimal Stopping"

September 12-16, 2010, Petrozavodsk, Russia .

1. Introduction

The time may be discrete or continuous.

We consider a time-homogeneous Markov process $Z = (Z_t)_{t\geq 0}$ taking values in $X \bigcup e$, where $(X, \mathcal{B})$ is a measurable space and e is an absorbing state.

1) $\rho(z) \geq 0$ — killing intensity.

2) $g(z)$ — payoff function, $g(e) = 0$.

3) $c(z)$ — cost of observation, $c(e) = 0$.

$$V(z,\tau) = E_z\left[g(Z_\tau) - \int_0^\tau c(Z_s)ds\right], \qquad V(z) = \sup_\tau V(z,\tau).$$

In discrete time we have $\sum_0^{\tau-1}$ instead of integral.

We suppose that the expectation is well–defined for all τ.

If $\rho(z) = constant$ then it is equivalent to the problem with discounting coefficient β where $\beta = \rho$ in continuous time or $\beta = 1 - \rho$ in discrete time.

C — subset of X,

$\tau_C = \inf\{t : Z_t \notin C\}$.

$g_C(z) = V(z, \tau_C) = E_z\left[g(Z_{\tau_C}) - \int_0^{\tau_C} c(Z_s)ds\right], \quad g_C(z) = g(z)$ if $z \notin C$.

Lemma 1. If $g_C(z) > g(z)$ for all $z \in C$ then the problem with payoff function $g_C(z)$ has the same value function as the problem with payoff function $g(z)$.

Proof. It follows from $g_C(z) \geq g(z)$ that $V_C(z) \geq V(z)$. From the other side for any τ we have $V_C(z, \tau) = V(z, \tau')$ where $\tau' = \inf\{t : t \geq \tau, Z_t \notin C\}$.

2. Discrete time

.

$\mathcal{P}$ — transition operator, $\mathcal{P}f(z) = E_z f(Z_1)$.

$\mathcal{T}f(z) = -c(z) + \mathcal{P}f(z)$ — revaluation operator

Theorem 1. a) *The function* $V(z)$ *is a minimal solution of the optimality equation (Bellman equation)*

$$V(z) = \max[g(z), \mathcal{T}V(z)]. \qquad (1)$$

b) *If* $\mathbf{P}_z[\tau^* < \infty] = 1$ *for all* $z \in X$*, where* $\tau^* = \inf\{n \geq 0 : \; Z_n \in D^*\}$, $D^* = \{z : V(z) = g(z)\}$*, then the stopping time* τ^* *is optimal one and* $\tau^* \leq \tau'$ $\mathbf{P}_z$*-a.s. for any* z *and any optimal stopping time* τ'*.*

c) *For the sequence* $\tilde{V}^{(0)}(z) = g(z)$, $\tilde{V}^{(k+1)}(z) = \max[g(z), \mathcal{T}\tilde{V}^{(k)}(z)]$ *the following relation holds* $\tilde{V}^{(k)} \uparrow V$.

$D^* = \{z : V(z) = g(z)\}$ - stopping set

$C^* = X \setminus D^* = \{z : V(z) > g(z)\}$ — continuation set

c) *For the sequence* $\tilde{V}^{(0)}(z) = g(z),\ \tilde{V}^{(k+1)}(z) = \max[g(z), \mathcal{T}\tilde{V}^{(k)}(z)]$ *the following relation holds* $\tilde{V}^{(k)} \uparrow V$.

It is said often that statement c) offers a constructive method for finding the value function $V(z)$ (see, for example, [2], p. 19).
Nevertheless, if $P_z[\tau^* > a] > 0$ for some $z \in X$ and any $a < \infty$ then $\tilde{V}_k(z) \leq \tilde{V}_{k+1}(z) < V(z)$ for this z and all k.

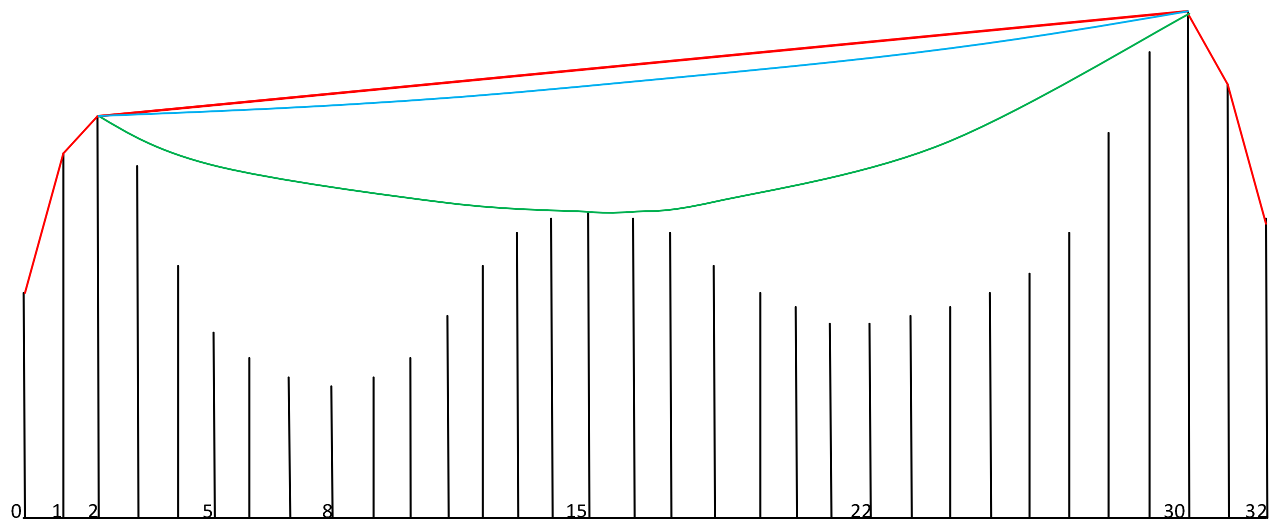
Example 1. Random walk on entire points of interval $[0, 32]$. Absorbtion at 0 and 32. Symmetric Bernoully at all other points. $c(z) = 0$. $g(2) = 11, g(8) = 4, g(15) = 8, g(22) = 5, g(30) = 14$.

The function $V(z)$ is depicted by red.

$\tilde{V}^{(k+1)}(z) = \max[g(z), (\tilde{V}^{(k)}(z-1) + \tilde{V}^{(k)}(z+1))/2]$.

Function $\tilde{V}^{(53)}(x)-$ green at the points where it does not coincide with $g(x)$.

The function $\tilde{V}^{(350)}(x)$ is depicted by blue. Here $\tilde{V}^{(350)}(16) = 11,77$, and $V(16) = 12,5$. So, even after 350 iteration the approximation error is $5,84\%$.



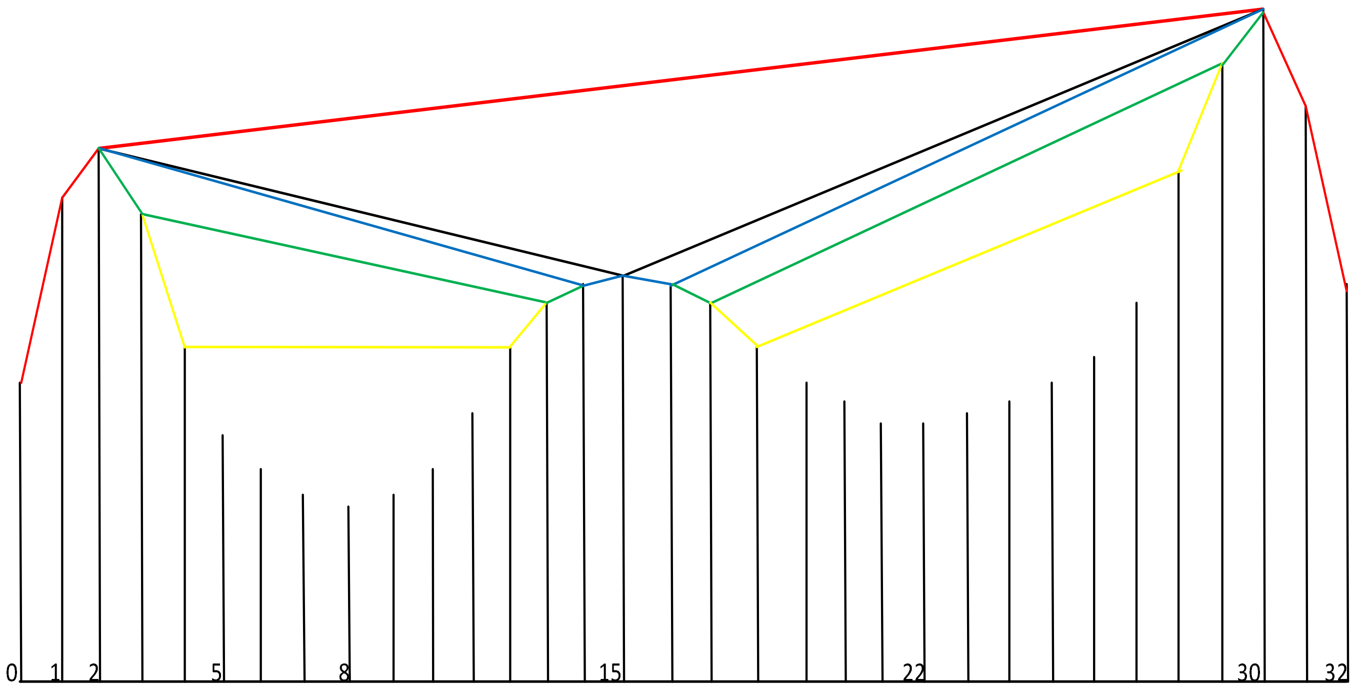
Condition A. Functions $g(z)$ and $c(z)$ are bounded and there exists $n_0 > 0$ such that $\mathbf{P}_z\{Z_{n_0} = e\} \geq 1 - \beta > 0$ for any $z \in X$.

Lemma 2. If Condition **A** is fulfilled and $C = \{z : \ Tg(z) > g(z)\}$ then $g_C(z) > g(z)$ for all $z \in C$ and the problem with payoff function $g_C(z)$ has the same value function as the initial problem with payoff function $g(z)$.

Sonin's State Elimination Algorithm. Finite state space

Sequentially we get the nondecreasing sequence of sets C_k which converges to the continuation set and the corresponding sequence of modified reward functions $g_k(z)$ which converges nondecreasingly to the value function of the initial problem.

Example 2. The same optimal stopping problem as in the Example 1.



The function $g_1(z)$ is depicted by yellow at the points where it does not coincide with $g(x)$. The function $g_2(x)$ is depicted respectively by green, the function $g_3(x)$ is depicted respectively by blue, the function $g_4(x)$ is depicted respectively by black, the function $g_5(x)$ is depicted respectively by red. So, after five iteration we got the value function and the stopping set.

3. One-dimensional diffusion

The general theory of the optimal stopping and methods of constructing the value function can be found, for example, in Peskir, Shiryaev (2006).

One-dimensional diffusion Dayanik, Karatzas (2003), Salminen (1985).

Time-homogeneous Markov process $Z = (Z_t)_{t \geq 0}$ taking values in $X \bigcup e$, where e is an absorbing state and $X = (a, b)$ is a measurable space and.

1) $\rho(z) \geq 0$ — killing intensity,

2) $g(z)$ — payoff function, $g(e) = 0$,

3) $c(z)$ — cost of observation, $c(e) = 0$,

4) $\sigma(z) \geq 0$ — diffusion coefficient,

5) $m(z)$ — drift coefficient,.

6) $a = z_0 < z_1 < \ldots < z_k < z_{k+1} = b$,

7) $\rho(z), c(z), m(z), \sigma(z), g(z), g'(z), g''(z)$ continuous on $(z_i, z_{i+1}),\ i = 0, \ldots k$,

8) $0 \leq \alpha_i < 1$ — reflection with probability α_i at point $z_i,\ i = 1, \ldots k$.

$$V(z,\tau) = E_z\left[g(Z_\tau) - \int_0^\tau c(Z_s)ds\right], \qquad V(z) = \sup_\tau V(z,\tau).$$

$a < c < d < b$,

$\tau_{(c,d)} = \inf\{t : Z_t \notin (c,d)\}$.

$g_{(c,d)}(z) = V(z, \tau_{(c,d)}) = E_z\left[g(Z_{\tau_{(c,d)}}) - \int_0^{\tau_{(c,d)}} c(Z_s)ds\right]$,

$g_{(c,d)}(z) = g(z)$ if $z \notin (c,d)$.

$Lg_{(c,d)}(z) := \frac{\sigma^2(z)}{2}\frac{d^2}{dz^2}g_{(c,d)}(z) + m(z)\frac{d}{dz}g_{(c,d)}(z) - \rho(z)g_{(c,d)}(z) - c(z) = 0$
for $z \in (c,d)$, $g_{(c,d)}(c) = g(c), g_{(c,d)}(d) = g(d)$,
if $z_i \in (c,d)$ then $(1+\alpha_i)g'_{+(c,d)}(z_i) - (1-\alpha_i)g'_{-(c,d)}(z_i) = 0$.

9) $Lg(z)$ does not change sign on (z_i, z_{i+1}), $i = 0, \ldots k$.

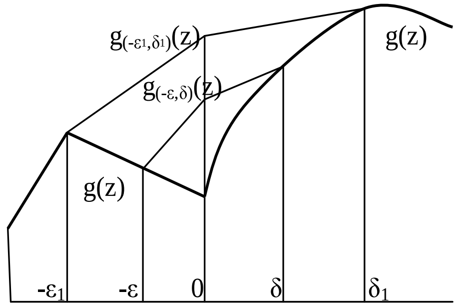
10) $g(z)$ is upper semi-continuous, i. e. $g(z) \geq \limsup_{z \to z_i} g(z)$, $i = 1, \ldots, k$.

Lemma 3. *If* $g(z_i) > \lim_{z\downarrow z_i} g(z)$ *then there exists* $\varepsilon \in (z_i, z_{i+1})$ *such that* $g_{(z_i,\varepsilon)}(z) > g(z)$ *for* $z \in (z_i, \varepsilon)$. *The same situation holds for the points where* $g(z_i) > \lim_{z\uparrow a} g(z)$.

Lemma 4. *If* $Lg(z) > 0$ *for* $z \in (z_i, z_{i+1})$ *then* $g_{(z_i,z_{i+1})}(z) > g(z)$ *for* $z \in (z_i, z_{i+1}),\ i = 1, \dots, k-1$.

Lemma 5. *If* $(1+\alpha_i)g'_+(z_i) - (1-\alpha_i)g'_-(z_i) > 0$ *then there exist* $\varepsilon \in (z_{i-1}, z_i)$ *and* $\delta \in (z_i, z_{i+1})$ *such that* $g_{(\varepsilon,\delta)}(z) > g(z)$ *for* $z \in (\varepsilon, \delta)$ *and* $g'_{+(\varepsilon,\delta)}(\varepsilon) > g'_{-(\varepsilon,\delta)}(\varepsilon)$, $g'_{+(\varepsilon,\delta)}(\delta) > g'_{-(\varepsilon,\delta)}(z)$.

Lemma 6. *If* $g(z)$ *is continuous,* $Lg(z) \le 0$ *for all points of continuity and* $(1+\alpha_i)g'_+(z_i) - (1-\alpha_i)g'_-(z_i) \le 0, \quad i = 1, \dots, k,$ *then* $V(z) = g(z)$.



Remark 3. One can say that an interval (c, d) in the problem with a smooth $g(z)$ is a smooth fitting interval if the function $g_{(c,d)}(z)$ has continuous derivative at points c and d. It can happen that such interval has no relation to the set C_* and one needs to use a verification theorem. In the proposed procedure we do not need to use a verification theorem.

4. Some examples

11 examples in Dayanik, Karatzas (2003).

Example 3. Geometric Brownian motion Z_t on $[0;\infty]$ with parameters (m,σ), killing intensity ρ and $g(z)=max[l,z]$ (Guo and Shepp (2001)).

$$Lf(z):=\left(\sigma^2z^2/2\right)f''(z)+mzf'(z)-\rho f(z).$$

Let $\kappa_+>0$ and $\kappa_-<0$ be the solutions of $\sigma^2\kappa^2-\left(\sigma^2-2m\right)\kappa-2\lambda=0.$

Then $g_{(c,d)}(z)=C_1z^{\kappa_+}+C_2z^{\kappa_-}$, where $C_1,\ C_2$ are chosen from the conditions $g_{(c,d)}(c)=g(c)$, $g_{(c,d)}(d)=g(d)$.

If $m>\rho$ then $g_{(c,d)}(z)\to+\infty$ and consequently $V(z)=+\infty$.

The case $m=\rho$ will be considered in Example 4.

Let now $m<\rho$.

Since $Lg(z)=-(\rho-m)z<0$ for $z>l$ and , $Lg(z)=-\rho<0$ for $0<z<l$ the only suspicious point is the point $z=1$.

Let $0 < c < l < d$. If $l - c$ and $d - l$ are small then $g_{(c,d)}(z) > g(z)$ for $z \in (c, d)$ and $g'_{+(c,d)}(c) > g'_{-(c,d)}(c) = 0$, $1 = g'_{+(c,d)}(d) > g'_{-(c,d)}(d) = 0$.

We can decrease c and increase d till the vakues $c^*,\ d^*$ for which

$g'_{+(c^*,d^*)}(c^*) = g'_{-(c^*,d^*)}(c^*) = 0$, $1 = g'_{+(c^*,d^*)}(d^*) = g'_{-(c^*,d^*)}(d^*) = 0$.

Example 4. Geometric Brownian motion Z_t on $(0;\infty]$ with parameters (m,σ), killing intensity m and $g(z)=(max[l,z]-K)^+$ (Guo and Shepp (2001)).

$$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) + mzf'(z) - mf(z).$$

Then $g_{(c,d)}(z) = C_1 z + C_2 z^\kappa$, where $\kappa = 2m/\sigma^2$ and $C_1,\ C_2$ are chosen from the conditions $g_{(c,d)}(c) = g(c)$, $g_{(c,d)}(d) = g(d)$.

Since $Lg(z) = -m(l-K) < 0$ for $0 < z < l$ and , $Lg(z) = mK$ for $z > l$ we have that if $c = l < d$ then $g_{(c,d)}(z) > g(z)$ for $z \in (c,d)$ and $g'_{+(c,d)}(c) > g'_{-(c,d)}(c) = 0$, $1 = g'_{+(c,d)}(d) > g'_{-(c,d)}(d) = 0$.

We can decrease c and increase d till the values $c^*,\ d^* = \infty$ for which

$g'_{+(c^*,d^*)}(c^*) = g'_{-(c^*,d^*)}(c^*) = 0.$

Example 5. Brownian motion Z_t with parameters $(m, 1)$, killing intensity ρ and $g(z) = 1$ for $z \le 0$ and $g(z) = 2$ for $g(z) = 1$ (Salminen (1985)).

$Lf(z) := f''(z) + mf'(z) - \rho f(z)$.

Let $\gamma_+ > 0$ and $\gamma_- < 0$ be the solutions of $\gamma^2 - m\gamma - \rho = 0$.

Then $g_{(c,d)}(z) = C_1 e^{z\gamma_+} + C_2 e^{z\gamma_-}$, where $C_1,\ C_2$ are chosen from the conditions $g_{(c,d)}(c) = g(c)$, $g_{(c,d)}(d) = g(d)$.

$Lg(z) = -m < 0$ for $-\rho < 0$ and $Lg(z) = -2\rho$ for $z > 0$. Chose $c < 0, |c|$ small, and $d \downarrow 0$. Then $g_{(c,0+)}(z) > g(z)$ for $z \in (c, 0)$ and $g'_{+(c,0+)}(c) > g'_{-(c,0+)}(c) = 0$,

We can decrease c till the value c^* for which

$g'_{+(c^*,0+)}(c^*) = g'_{-(c^*,0+)}(c^*) = 0$.

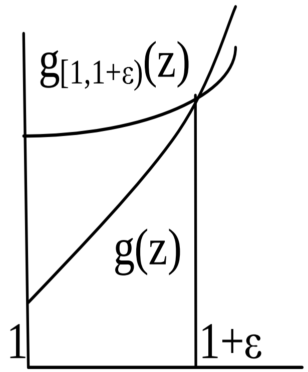
Example 6. Geometric Brownian motion Z_t on $[1;\infty]$ with parameters $(-m,\sigma)$, killing intensity ρ, reflection at the point 1 and with functional $E_z\left[x_\tau\right]$.
This example corresponds to the Russian option (see [2], Section 26).

$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) - mzg''(z) - \rho f(z).$
Since in this case $g(z) = z$ and $Lg(z) = -(m+\rho)z < 0$, the only suspicious point is the point $z = 1$.

Let $\kappa_+ > 0$ and $\kappa_- < 0$ be the solutions of $\sigma^2\kappa^2 - \left(\sigma^2 + 2m\right)\kappa - 2\lambda = 0$.
The reflection corresponds to the condition $g'_{[1,a)}(1) = 0$ and

$$g_{[1,a)}(z) := \frac{g(a)\left(\kappa_+ z^{\kappa_-} - \kappa_- z^{\kappa_+}\right)}{\kappa_+ a^{\kappa_-} - \kappa_- a^{\kappa_+}} \text{ for } z \in [1,a)\,.$$

If $a-1$ is small then $g'_{[1,a)}(a) < g'(a)$ and $g_{[1,a)}(z) > g(z)$ for $z \in [1,a)$.
The optimal value a^* can be found as earlier from the condition
$a^* = \{\inf a : g'_{[1,a)}(a) \geq g'(a) \equiv 1\}$.


g[1,1+ε)(z)
g(z)
1
1+ε

Список литературы

[1] S. Dayanik, I. Karatzas, *On the optimal stopping problem for one-dimensional diffusions.* Stochastic Processes & Applications, 107, 2003, pp. 173-212.

[2] G. Peskir, A.N. Shiryaev, *Optimal stopping and free-boundary problems*, Birkhauser, Basel, 2006.

[3] Presman E. L., *On Sonin's algorithm for solution of the optimal stopping problem*, Proceedings of the Fourth international conference on control problems (January 26—30, 2009). — M.: Institute of Control Sciences, 2009, pp. 300–309.

[4] Presman E. L., Sonin I. M., *On one problem of optimal stopping for random variables given on a Markov chain*, Theory of Probab. and its Applic., v. 54(3) (2009).

[5] P. Salminen, *Optimal stopping of one-dimensional diffusions*, Mathematische Nachrichten, 124, 1985, pp. 85-101.

[6] I.M. Sonin, *Two simple theorems in the problems of optimal stopping*, In: Proc. 8th INFORMS Appl. Prob. Conf., Atlanta, Georgia, 1995, p. 27.

[7] I.M. Sonin, *The elimination algorithm for the problem of optimal stopping*, Math. Meth. of Oper. Res., 49 (1999), pp.111-123.

[8] I.M. Sonin, *The state reduction and related algorithms and their applications to the study of Markov chains, graph theory and the optimal stopping problem*, Advances in Math., 145 (1999), pp. 159-188.

[9] I.M. Sonin, *Optimal stopping of Markov chains and recursive solution of Poisson and Bellman equations*, In: From Stochastic Calculus to Mathematical Finance. The Shiryaev Festschrift, Springer-Verlag, 2006, pp. 609-621.