Differential Games with Random Terminal Time

Ekaterina Shevkoplyas

Faculty of Applied Mathematics and Processes of Control,
St. Petersburg State University,
Russia

SOS'2010
Petrozavodsk, 2010

Outline

1. Game duration. 3 concepts
2. Differential games with random duration
3. Example of differential game with random duration
4. Time-consistency problem

Game duration. Main formulations

1. $t \in [t_0; T]$; duration $(T - t_0)$.
2. Infinite time horizon: $t \in [t_0, \infty)$; duration is ∞.

1. Integral payoff of player i:

$$K_i(x_0, u_1, \ldots, u_n) = \int_{t_0}^{T} h_i(x, \tau, u) d\tau.$$

2. Integral payoff of player i:

$$K_i(x_0, u_1, \ldots, u_n) = \int_{t_0}^{\infty} h_i(x, \tau, u) \cdot e^{-(\tau - t_0)} d\tau.$$

Game duration. Main formulations

1. $t \in [t_0; T]$; duration $(T - t_0)$.
2. Infinite time horizon:
   $t \in [t_0, \infty)$; duration is ∞.

1. Integral payoff of player i:

$$K_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{T} h_i(x, \tau, u) d\tau.$$

2. Integral payoff of player i:

$$K_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{\infty} h_i(x, \tau, u) \cdot e^{-(\tau - t_0)} d\tau.$$

Random Duration

$t \in [t_o; T]$, but T is a random variable with distribution function $F(t)$, $\quad t \in [t_0, \infty)$! Then the duration $(T - t_0)$ is random!

References

1. Petrosjan, L.A. and Murzov, N.V. Game-theoretic problems of mechanics, 1966.
2. Yaari, M.E. Uncertain Lifetime, Life Insurance, and the Theory of the Consumer, 1965.

Integral payoff of player i:

$$K_i(x_0, u_1, \ldots, u_n) = E\left(\int_{t_0}^{T} h_i(x, \tau, u)d\tau\right) =$$

$$= \int_{t_0}^{\infty}\int_{t_0}^{t} h_i(x, \tau, u)d\tau dF(t).$$

Let $\exists \quad f(t) = F'(t)$. Then $K_i = \int_{t_0}^{\infty}\int_{t_0}^{t} h_i(x, \tau, u)d\tau f(t)dt$.

The simplification of integral functional

Integration by parts: [Burness, 1976], [Boukas, Haurie and Michel, 1990], [Chang, 2004]

$$\int_{t_0}^{\infty} \int_{t_0}^{t} h(x, u, \tau)\, d\tau\, f(t) dt. \tag{1}$$

Let $t_0 = 0$. Let us denote $h(x, u, \tau) = h(\tau)$.

$$\int_{0}^{\infty} \int_{0}^{t} h(\tau)\, d\tau\, f(t) dt. \tag{2}$$

Define

$$g(t, \tau) = f(t)h(\tau) \cdot \chi_{\{\tau \leq t\}} = \begin{cases} f(t)h(\tau), & \tau \leq t; \\ 0, & \tau > t. \end{cases} \tag{3}$$

The simplification of integral functional

$$\int_0^{+\infty} dt \int_0^t f(t)h(\tau)d\tau = \int_0^{+\infty} dt \int_0^{+\infty} g(t,\tau)d\tau \overset{!!}{=} \iint\limits_{[0,+\infty)\times[0,+\infty)} g(t,\tau)dtd\tau \overset{!!}{=}$$

$$\overset{!!}{=} \int_0^{+\infty} d\tau \int_0^{+\infty} g(t,\tau)dt = \int_0^{+\infty} d\tau \int_\tau^{+\infty} f(t)h(\tau)dt =$$

$$= \int_0^{+\infty} (1 - F(\tau))h(\tau)d\tau. \quad (4)$$

Thus, the integral payoff is

$$K_i(x_0, u_1, \ldots, u_n) = \int_{t_0}^{+\infty} (1 - F(\tau))h_i(x, \tau, u)d\tau.$$

Game $\Gamma(x_0)$

Payoff of player i: integral payoff + terminal payoff

$$K_i(x_0, u_1, \dots, u_n) = \int_{t_0}^{+\infty} (1 - F(\tau)) h_i(x, \tau, u) d\tau + \tag{5}$$

$$+ \int_{t_0}^{+\infty} f(\tau) H_i(x, \tau, u) d\tau, \qquad i = 1, \dots, n. \tag{6}$$

$$\dot{x} = g(x, u_1, \dots, u_n), \quad x \in R^n, \ u_i \in U \subseteq \text{comp}\, R^l, \tag{7}$$
$$x(t_0) = x_0.$$

Conditional distribution function

$(1 - F(\vartheta))$ is the probability to start $\Gamma(x(\vartheta))$.
The conditional distribution function $F_\vartheta(t)$:

$$F_\vartheta(t) = \frac{F(t) - F(\vartheta)}{1 - F(\vartheta)}, \quad t \in [\vartheta, \infty). \tag{8}$$

The conditional density function $f_\vartheta(t)$:

$$f_\vartheta(t) = \frac{f(t)}{1 - F(\vartheta)}, \ t \in [\vartheta, \infty). \tag{9}$$

Subgame $\Gamma(x(\vartheta))$

$(1 - F(\vartheta))$ is the probability to start $\Gamma(x(\vartheta))$.
Subgame $\Gamma(x(\vartheta))$:
Payoff

$$K_i(x, \vartheta, u_1, \dots, u_n) = \frac{1}{1 - F(\vartheta)} \int_{\vartheta}^{\infty} (1 - F(\tau)) h_i(x, \tau, u) d\tau + \frac{1}{1 - F(\vartheta)} \int_{\vartheta}^{\infty} H_i(x, \tau, u) f(\tau) d\tau. \quad (10)$$

$$\dot{x} = g(x, u_1, \dots, u_n), \quad x \in R^n,\ u_i \in U \subseteq \operatorname{comp} R^l, \qquad (11)$$
$$x(\vartheta) = x.$$

Hamilton-Jacobi-Bellman equation

Maximization problem:

$$\frac{1}{1-F(t)}\int_t^{\infty}\Big(h(x,u,s)(1-F(s))+H(x,u,s)f(s)\Big)ds.$$

$$\dot{x}=g(x,u)$$
$$x(t)=x.$$

Let W be Bellman function for this problem.
Let us consider the optimization problem

$$\int_t^{\infty}\Big(h(x,u,s)(1-F(s))+H(x,u,s)f(s)\Big)ds.$$

$$\dot{x}=g(x,u)$$
$$x(t)=x.$$

Let $\bar{W}$ be Bellman function for this problem.

Hamilton-Jacobi-Bellman equation

Obviously,

$$\bar{W}() = W() \cdot (1 - F(t)). \tag{12}$$

Then

$$\frac{\partial \bar{W}}{\partial t} = -f(t)W + (1 - F(t))\frac{\partial W}{\partial t}; \tag{13}$$

$$\frac{\partial \bar{W}}{\partial x} = (1 - F(t))\frac{\partial W}{\partial x}. \tag{14}$$

For the problem with Bellman function $\bar{W}$ we have the standard HJB equation:

$$\frac{\partial \bar{W}}{\partial t} + \max_u \left(h(x,u,t)(1 - F(t)) + H(x,u,t)f(t) + \frac{\partial \bar{W}}{\partial x} g(x,u) \right) = 0.$$

Using (12), (13), (14) we get HJB equation for the problem with random duration:

$$\frac{f(t)}{1 - F(t)} W = \frac{\partial W}{\partial t} + \max_u \left(h(x,u,t) + \frac{f(t)}{1 - F(t)} H(x,u,t) + \frac{\partial W}{\partial x} g(x,u) \right).$$

Hamilton-Jacobi-Bellman equation. Analysis

$$\frac{f(t)}{1-F(t)}W = \frac{\partial W}{\partial t} + \max_u \Big(h(x,u,t) + \frac{f(t)}{1-F(t)}H(x,u,t) + \frac{\partial W}{\partial x} g(x,u) \Big). \quad (15)$$

Petrosyan L.A., Murzov N.V., 1966 — Bellman-Isaacs equation for the differential pursuit game with terminal payoff at random final time instant

Shevkoplyas E., 2004 (without simplification to standard DP problem)

Boukas, E. K. and Haurie, A. and Michel, P., 1990 —Bellman equation for optimal control problem with a random stopping time

Chang, F.R., 2004 — Recursive utility function (Stochastic Optimization in Continuous Time).

Hamilton-Jacobi-Bellman equation. Analysis

$$\frac{f(t)}{1-F(t)}W = \frac{\partial W}{\partial t} + \max_u\Big(h(x,u,t) + \frac{f(t)}{1-F(t)}H(x,u,t) + \frac{\partial W}{\partial x}g(x,u)\Big). \quad (15)$$

Petrosyan L.A., Murzov N.V., 1966 — Bellman-Isaacs equation for the differential pursuit game with terminal payoff at random final time instant

Shevkoplyas E., 2004 (without simplification to standard DP problem)

Boukas, E. K. and Haurie, A. and Michel, P., 1990 —Bellman equation for optimal control problem with a random stopping time

Chang, F.R., 2004 — Recursive utility function (Stochastic Optimization in Continuous Time).

Game theory. Reliability theory

Game theory: T – time instant (random) when the game ends

$\Downarrow$

Reliability theory: T – time of failure of the system

Hazard function (failure rate):

$$\lambda(t) = \frac{f(t)}{1 - F(t)}. \qquad (16)$$

Game theory. Reliability theory

Game theory: T – time instant (random) when the game ends

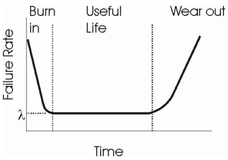
$\Downarrow$

Reliability theory: T – time of failure of the system

Hazard function (failure rate):

$$\lambda(t) = \frac{f(t)}{1 - F(t)}. \qquad (16)$$

Hazard function $\lambda(t)$



The life circle and the hazard function $\lambda(t)$:

1. Burn- in ("infant"). Infant mortality or early failures. Decreasing failure rate $\lambda(t)$.
2. Normal life (useful life,"adult"). Random failures. Constant failure rate $\lambda(t) = \lambda$.
3. End of life (wear-out). Increasing failure rate $\lambda(t)$.

Hazard function $\lambda(t)$ + HJB equation

From (15) we get

$$\lambda(t)W = \frac{\partial W}{\partial t} + \max_u \left(h(x,u,t) + \lambda(t)H(x,u,t) + \frac{\partial W}{\partial x} g(x,u) \right). \quad (17)$$

Moreover, $(1 - F(t)) = e^{-\int_{t_o}^{t} \lambda(\tau)d\tau}$. Then

$$K_i(x_0, u_1, \ldots, u_n) = \int_{t_0}^{+\infty} e^{-\int_{t_o}^{\tau} \lambda(s)ds} h_i(x, \tau, u)d\tau + \quad (18)$$
$$+ \int_{t_0}^{+\infty} f(\tau)H_i(x, \tau, u)d\tau, \qquad i = 1, \ldots, n.$$

Problem with random terminal time is an extension of the problem with constant discounting!

Constant hazard rate

Hazard rate is constant only for exponential distribution of failure time.

$$\frac{f(t)}{1-F(t)} = \lambda(t) = \lambda = const$$

only for density probability distribution

$$f(t) = \lambda e^{-\lambda(t-t_0)}, \qquad t > t_0.$$

Then

the problem with random exponential final time is equivalent to problem with constant discounting

"End of life"distributions

Reliability theory

For technical systems:

- Exponential
- Weibull
- Normal
- Logarithmic-normal
- Gamma

Actuarial mathematics, gerontology

For biological systems:

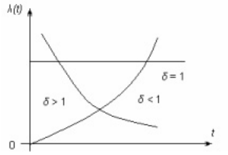
- Gompertz-Makeham
- Weibull

[Cox, Lewis, 1966]: time intervals between accidents in coal-pits and even strike actions of coalminers are well defined by Weibull

Weibull distribution

Hazard rate

$$\lambda(t) = \lambda\delta t^{\delta-1}; \tag{19}$$
$$t \geq 0;\ \lambda > 0;\ \delta > 0.$$



$\delta < 1$ Burn-in. $\lambda(t)$ is decreasing.

$\delta = 1$ Normal life. $\lambda(t)$ is constant λ. Weibull distribution equivalents to exponential distribution.

$\delta > 1$ Wear-out. $\lambda(t)$ is increasing. $\delta = 2$ Raleigh distribution

Model of non-renewable resource extraction

E.J. Dockner, S. Jorgensen, N. van Long, G. Sorger, 2000

Players: firms, countries,...
The set of the players: $I = \{1, 2, ..., n\}$.
Let $x(t)$ be the stock of the nonrenewable resource such as an oil field.
Let $c_i(t)$ be player i's rate of resource extraction at time t.
The transition equation:

$$\dot{x}(t) = -\sum_{i=1}^{n} c_i(t); \tag{20}$$

$$\lim_{t->\infty} x(t) \geq 0; \tag{21}$$

$$x(t_0) = x_0. \tag{22}$$

Model of non-renewable resource extraction

Each player i has a utility function $h(c_i)$, defined for all $c_i > 0$. The utility function:

$$h(c_i) = ln(c_i). \tag{23}$$

Let $t_0 = 0$. T is a random variable with Weibull distribution.
Expected payoff of the player i:

$$K_i(x_0, u_1, \ldots, u_n) = \int_0^\infty (1 - F(\tau))h_i(\cdot)d\tau, \tag{24}$$

$$F(t) = 1 - e^{-\lambda t^\delta}.$$

Model of non-renewable resource extraction

Each player i has a utility function $h(c_i)$, defined for all $c_i > 0$. The utility function:

$$h(c_i) = ln(c_i). \tag{23}$$

Let $t_0 = 0$. T is a random variable with Weibull distribution. Expected payoff of the player i:

$$K_i(x_0, u_1, \ldots, u_n) = \int_0^\infty (1 - F(\tau))h_i(\cdot)d\tau, \tag{24}$$

$$F(t) = 1 - e^{-\lambda t^\delta}.$$

Solution

$$c_i^I = \frac{e^{-\lambda(t)t}}{n\int\limits_t^{\infty} e^{-\lambda(s)s}ds}x. \quad (25)$$

For the case of normal exploitation stage ($\lambda(t) = \lambda$) or exponential distribution of final game time, we get the result for optimal extraction rule in explicit form:

$$c_i^I = \frac{\lambda}{n}x, \qquad i = 1, \dots, n. \quad (26)$$

Finally, we have optimal trajectory and optimal controls

$$x^I(t) = x_o * e^{-\lambda(t-t_0)}; \quad (27)$$
$$c_i^I(t) = \frac{x_0\lambda}{n}e^{-\lambda(t-t_0)},$$

Solution

If $\delta = 2$ (Raileigh distribution, wear-out stage), we get the following extraction rule from (25):

$$c_i^I = \frac{x \cdot e^{-2\lambda t^2}}{n \int_t^\infty e^{-2\lambda s^2} ds}. \quad (28)$$

Then the optimal solution is as follows

$$c_i^I = \frac{2\sqrt{2}\sqrt{\lambda} \cdot e^{-2\lambda t^2}}{n(1 - erf(\sqrt{2\lambda})t)} x, \qquad \text{where} \quad (29)$$

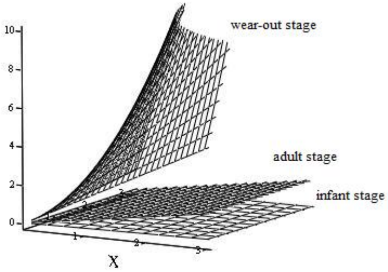
$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds.$$

For a case of $\delta = \frac{1}{2}$ (burn-in period) we get the extraction rule

$$c_i^I = \frac{x \cdot e^{-\frac{\lambda}{2} t^{1/2}}}{n \int_t^\infty e^{-\frac{\lambda}{2} s^{1/2}} ds}. \quad (30)$$

Finally, we get

$$c_i^I = \frac{\lambda^2}{n4(\lambda\sqrt{t} + 2)} x. \quad (31)$$



The optimal rate of resource extraction at an initial stage is low than for an "adult"stage (it is equivalent for accuracy of players at an early stage), but the highest rate is for wear-out stage. Of course, the resource stock $x(t)$ at the last stage tends to zero, but the optimal behavior according to our simple model is to excavate "like grim death".

Time-consistency problem. Review

Time-consistency problem in resources and environmental economics: L. A. Petrosjan, David W.K. Yeung, G. Zaccour, S. Jorgensen, M. Breton, G. Martin-Herran, V.V. Zakharov, V.V. Mazalov, N.A. Zenkevich etc.

- Leon A. Petrosyan (1977)
- Prescribed game duration $T - t_0$ (finite time horizon)

Imputation Distribution Procedure (IDP): $\beta(\tau) = (\beta_1(\tau), \ldots, \beta_n(\tau))$, $\tau \in [t_0, T]$ such that

$$Sh_i = Sh_i\,(x_0, t_0) = \int_{t_0}^{T} \beta_i(\tau) d\tau. \qquad (32)$$

Time-consistency problem. Review

The vector $\beta(t)$ is a time-consistent IDP if at $(x^*(t), t)$, $\forall t \in [0, \infty)$ the following condition holds:

$$Sh_i^t = Sh_i\left(x^*(t), t\right) = \int_t^T \beta_i(\tau) d\tau$$

or

$$\beta_i(t) = -(Sh_i^t)_t^{'}. \qquad (33)$$

From (33) we get

$$Sh_i = \int_{t_0}^t \beta_i(\tau) d\tau + Sh_i^t. \qquad (34)$$

Time-consistency problem. Random Duration

$$\bar{\xi}_i = \int_{t_0}^{\vartheta} (1 - F(\tau))\gamma_i(\tau)d\tau + (1 - F(\vartheta))\bar{\xi}_i^{\vartheta}. \quad (35)$$

Differentiating (35) with respect to t we get the formula for IDP:

$$\gamma_i(\vartheta) = \frac{F'(\vartheta)}{1 - F(\vartheta)}\bar{\xi}_i^{\vartheta} - (\bar{\xi}_i^{\vartheta})'. \quad (36)$$

For the case of exponential distribution it covers the result of Petrosyan L., Zaccour G.(2003):

$$\gamma_i(\vartheta) = \rho\bar{\xi}_i^{\vartheta} - (\bar{\xi}_i^{\vartheta})'. \quad (37)$$

Time-consistency problem. Random Duration

$$\bar{\xi}_i = \int_{t_0}^{\vartheta} (1 - F(\tau))\gamma_i(\tau)d\tau + (1 - F(\vartheta))\bar{\xi}_i^{\vartheta}. \qquad (35)$$

Differentiating (35) with respect to t we get the formula for IDP:

$$\gamma_i(\vartheta) = \frac{F'(\vartheta)}{1 - F(\vartheta)}\bar{\xi}_i^{\vartheta} - (\bar{\xi}_i^{\vartheta})'. \qquad (36)$$

For the case of exponential distribution it covers the result of Petrosyan L., Zaccour G.(2003):

$$\gamma_i(\vartheta) = \rho\bar{\xi}_i^{\vartheta} - (\bar{\xi}_i^{\vartheta})'. \qquad (37)$$

Time-consistency problem. Example

$$Sh_i(x^I(t)) = \frac{V(I,x)}{n} = \frac{ln(x^I(t))}{\lambda} - \frac{1}{\lambda} - \frac{ln(n)}{\lambda} + \frac{ln(\lambda)}{\lambda} = \tag{38}$$
$$= \frac{ln(x_0)}{\lambda} - (t - t_0) - \frac{1}{\lambda} - \frac{ln(n)}{\lambda} + \frac{ln(\lambda)}{\lambda},$$

$$Sh_i(x_0) = \frac{V(I,x_0)}{n} = \frac{ln(x_0)}{\lambda} - \frac{1}{\lambda} - \frac{ln(n)}{\lambda} + \frac{ln(\lambda)}{\lambda}.$$

$$\gamma_i(\vartheta) = ln(x_0) - \lambda(\vartheta - t_0) + ln(\lambda) - ln(n). \tag{39}$$

References

- Petrosjan L .A., Shevkoplyas E. V. *Cooperative Differential Games with Random Duration*. // Vestnik SPbGU. –2000.– Ser. 1, Vol.4. – pp. 18- 23 (in Russian).
- Petrosjan L .A., Shevkoplyas E. V. *Cooperative Solutions for Games with Random Duration*. Game Theory and Applications, Volume IX. Nova Science Publishers, 2003, pp.125-139.
- Shevkoplyas, E.V. *The Hamilton-Jacobi-Bellman equation in differential games with random duration*, Upravlenie bolshimi systemami, IPU RAN, Moscow, 2009. (in Russian)