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A QUICKEST DETECTION PROBLEM

with an expensive cost of observations

§ 1. A WELL-KNOWN QUICKEST DETECTION MODEL:

(a) observations $X = (X_t)_{t \geq 0}$ obey the equation

$$dX_t = rI(\theta \leq t)\,dt + \sigma\,dB_t, \qquad X_0 = 0,$$

or

$$X_t = \begin{cases} \sigma B_t, & t < \theta, \\ r(t-\theta) + \sigma B_t, & t \geq \theta; \end{cases}$$

(b) Brownian motion B and random variable θ are independent, usually $\theta \sim \mathrm{Exp}(\pi, \lambda)$, i.e.,
$\mathrm{P}(\theta = 0) = \pi$, $\mathrm{P}(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$, $\lambda > 0$;

(c) parameters $\sigma > 0$, $r \in \mathbb{R}$, $\lambda > 0$ are known; $\pi \in [0,1]$.

NEW MODEL STOPPING + CONTROL:

observable process $X^h = (X_t^h)_{t \geq 0}$ obey the equation

$$dX_t^h(\omega) = rh_t(\omega) I(\theta(\omega) \leq t)\, dt + \sigma \sqrt{h_t(\omega)}\, dB_t(\omega)$$

with an $\mathcal{F}_t$-measurable $h_t = h_t(\omega) \in [0,1]$ (all objects are given on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P})$); B and θ are independent.

The pair (h, X^h) is called a **control system**. Here

$$h_t(\omega) = h_t(X^h(\omega))$$

is a **synthesis-control**, where the mapping $(t, x) \rightsquigarrow h_t(x)$ is $\mathcal{C}_t = \sigma\{x \colon x_s, s \leq t\}$-measurable.

The system $(h(x), X^h)$ is called **admissible** if stochastic differential equation

$$dX_t^h(\omega) = rh_t(X^h(\omega))I(\theta(\omega) \le t)\,dt + \sigma\sqrt{h_t(X^h)}\,dB_t(\omega)$$

has a solution * .

The admissible pairs (h, X) (where $X = X^h$) will be called **canoni-cal** ** **control system**.

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* weak or strong; the class of weak solutions is larger than the class of strong solutions; from point of view of real applications, it is better to have strong solutions; from point of view of distributional analysis of the problem, it is reasonable to operate with weak solutions.

** because $h_t(\omega) = h_t(X_s(\omega), s \le t)$.

§ 2. To formulate our problem we introduce stopping times $\tau = \tau(x)$, $x \in C = C[0,\infty)$, which play the role of the signal about appearing of a 'change-point' $\theta = \theta(\omega)$.

For the process $X = X^h$,

$\tau(X)$ denotes the composition $(\tau \circ X)(\omega)$,
i.e., random variable $\tau(X(\omega))$.

The set (h, X, τ) plays a key role in our formulation of the quickest detection problem. The pair (h, τ) is called a **strategy**.

With (h,τ) we relate the penalty function

$$G(h,\tau) = I(\tau(X) < \theta) + a(\tau(X) - \theta)^{+} + b\int_0^{\tau(X)} h_t(X)\,dt,$$

where θ is a random variable, $\theta \sim \text{Exp}(\pi,\lambda)$: $\mathsf{P}(\theta = 0) = \pi$, $\mathsf{P}(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$.

Denote by P_π the distribution of X under assumption $\mathsf{P}(\theta = 0) = \pi$. Note that $\text{Law}(B \,|\, \mathsf{P}_\pi)$ does not depend on π.

The value function of the "stopping-control" problem is defined by

$$V^*(\pi) = \inf_{(h,\tau)} \mathsf{E}_\pi G(h,\tau), \qquad \pi \in [0,1].$$

We want to find $V^*(\pi)$ and describe an optimal strategy (h^*,τ^*).

§ 3. SOME AUXILIARY PROPOSITIONS.

LEMMA 1. Function $V^* = V^*(\pi)$ is concave.

Proof. By formula of complete probability,

$$\mathsf{E}_\pi G(h,\tau) = \pi \mathsf{E}_\pi\left[\tau + \int_0^\tau h_t\,dt \,\middle|\, \theta = 0\right] \\ + (1-\pi)\mathsf{E}_\pi\left[I(\tau<\theta) + a(\tau-\theta)I(\tau>\theta) + b\int_0^\tau h_t\,dt \,\middle|\, \theta > 0\right].$$

None of expectations $\mathsf{E}_\pi(\,\cdot\,|\,\theta=0)$ and $\mathsf{E}_\pi(\,\cdot\,|\,\theta>0)$ depends on π. So, $\mathsf{E}_\pi G(h,\tau)$ is an affine function of π, and $V^*(\pi)$ as infimum of affine functions is a concave function. $\square$

In the sequel, an important role is played by **a priori** and **a posteriori probabilities** $(p_t)_{t\ge 0}$ and $(\pi_t^h)_{t\ge 0}$:

$$p_t = \mathsf{P}_\pi(\theta \le t) = \pi + (1-\pi)(1-e^{-\lambda t}),$$
$$\pi_t^h = \mathsf{P}_\pi(\theta \le t \,|\, \mathcal{F}_t^{X^h}), \quad \text{where } \mathcal{F}_t^{X^h} = \sigma(X_s^h, s \le t).$$

LEMMA 2. For each strategy (h,τ) we have

$$\mathsf{E}_\pi G(h,\tau) = \mathsf{E}_\pi \Big[(1-\pi_\tau^h) + a\int_0^\tau \pi_t^h\,dt + b\int_0^\tau h_t\,dt \Big].$$

The lemma follows from

$$\mathsf{E}_\pi I(\tau > \theta) = \mathsf{E}_\pi(1 - \pi_\tau^h)$$

and

$$\begin{aligned}
\mathsf{E}_\pi(\tau - \theta)^+ &= \mathsf{E}_\pi(\tau - \theta) I(\tau \ge \theta) = \mathsf{E}_\pi \int_0^\infty I(\theta \le t < \tau)\, dt \\
&= \mathsf{E}_\pi \int_0^\infty \mathsf{E}_\pi \Big[I(\theta \le t) I(t < \tau) \,|\, \mathcal{F}_t^{X^h} \Big]\, dt \\
&= \mathsf{E}_\pi \int_0^\infty I(t < \tau) \mathsf{E}_\pi \Big[I(\theta \le t) \,|\, \mathcal{F}_t^{X^h} \Big]\, dt \\
&= \mathsf{E}_\pi \int_0^\tau \pi_t^h \, dt,
\end{aligned}$$

where we used the fact that $\{\tau \le t\} \in \mathcal{F}_t^{X^h}$. $\square$

The a priori probability $(p_t)_{t\geq 0}$ solves the equation

$$dp_t = \lambda(1 - p_t)\,dt, \qquad t \geq 0,$$

with $p_0 = \pi$.

In the following lemmas we give stochastic differential equations for

$$\pi_t^h = \mathsf{P}(\theta \leq t \,|\, \mathcal{F}_t^{X^h}) \quad \text{and} \quad \varphi_t^h = \frac{\pi_t^h}{1 - \pi_t^h}.$$

We can assume that $\pi < 1$, since if $\pi = 1$, then $\pi_t^h = 1$ for all $t > 0$.

Let $\boxed{\mu^h_{t,u} = \text{Law}(X^h_s, s \le t \mid \theta = u)}$ (these measures don't depend on π

A special role belongs to the measures $\boxed{\mu^h_{t,0} \quad \text{and} \quad \mu^h_{t,t}}$ $(\mu^h_{t,t} = \mu^h_{t}$
$\forall u > t$, and also for $u = \infty$ when there is no "disorder" at all).

The Radon–Nikodým derivative $L^h_t = \dfrac{d\mu^h_{t,0}}{d\mu^h_{t,t}} \equiv \dfrac{d(\text{Law}(X^h_s, s \le t) \mid \theta = 0}{d(\text{Law}(X^h_s, s \le t) \mid \theta = t}$

is given by the formula

$$L^h_t = \exp\left\{ \int_0^t \frac{r}{\sigma^2}\, dX^h_s - \frac{1}{2} \int_0^t \frac{r^2}{\sigma^2} h_s\, ds \right\}.$$

By Itô's formula, $\quad dL^h_t = \dfrac{r}{\sigma^2}\, L^h_t\, dX^h_t. \quad$ Assuming that $\pi \neq 1$, put

$$\varphi^h_t = \frac{\pi^h_t}{1 - \pi^h_t}.$$

LEMMA 3. We have

$$\varphi_t^h = \varphi_0^h e^{\lambda t} L_t^h + \lambda \int_0^t e^{\lambda(t-u)} \frac{L_t^h}{L_u^h}\, du, \qquad d\varphi_t^h = \lambda(1+\varphi_t^h)\, dt + \frac{r}{\sigma^2}\, \varphi_t^h\, dX_t^h.$$

Proof. By Bayes' formula,

$$\varphi_t^h = \frac{\mathsf{P}_\pi(\theta \le t \,|\, \mathcal{F}_t^{X^h})}{\mathsf{P}_\pi(\theta > t \,|\, \mathcal{F}_t^{X^h})} = \frac{\pi \frac{d\mu_{t,0}}{d\mu_{t,\infty}} + (1-\pi) \int_0^t \lambda e^{-\lambda u} \frac{d\mu_{t,u}}{d\mu_{t,\infty}}\, du}{(1-\pi) \int_t^\infty \lambda e^{-\lambda u} \frac{d\mu_{t,u}}{d\mu_{t,\infty}}\, du}$$

$$= \frac{\pi}{1-\pi}\, e^{\lambda t} L_t^h + \lambda \int_0^t e^{\lambda(t-u)} \frac{L_t^h}{L_u^h}\, du,$$

where we used the formulas $\displaystyle \frac{d\mu_{t,u}}{d\mu_{t,\infty}} = \begin{cases} 1 & \text{for } u \ge t, \\ L_t^h / L_u^h & \text{for } u \le t. \end{cases}$ □

LEMMA 4. **(a)** The a posteriori probability $(\pi_t^h)_{t\geq 0}$ satisfies the stochastic differential equation

$$d\pi_t^h = \lambda(1-\pi_t^h)\,dt + \frac{r}{\sigma^2}\,\pi_t^h(1-\pi_t^h)\Big(dX_t^h - rh_t\pi_t\,dt\Big).$$

(b) The process $X^h = (X_t^h)_{t\geq 0}$ admits the innovation representation

$$dX_t^h = rh_t\pi_t^h\,dt + \sigma\sqrt{h_t}\,d\overline{B}_t^h, \qquad X_0^h = 0,$$

where $\overline{B} = (\overline{B}_t^h)_{t\geq 0}$ is a (standard) Brownian motion (with respect to $(\mathcal{F}_t^{X^h})_{t\geq 0}$).

(c) The process $\pi^h = (\pi_t^h)_{t\geq 0}$ admits the innovation representation

$$d\pi_t^h = \lambda(1-\pi_t^h)\,dt + \frac{r}{\sigma^2}\,\pi_t^h(1-\pi_t^h)\sqrt{h_t}\,d\overline{B}_t^h \quad \text{with } \pi_0^h = \pi.$$

Proof. (a) follows by Itô's formula from Bayesian representation of π_t^h.

(b) Note that

$$X_t^h - \int_0^t r\pi_s^h \, ds = \int_0^t rh_s \Big[I(\theta \le s) - \pi_s^h \Big] \, ds + \int_0^t \sigma \sqrt{h_s} \, dB_s.$$

The process on the right-hand side is a martingale with $\langle \cdot \rangle_t = \int_0^t \sigma^2 h_s \, ds$. The existence of innovation process (maybe on the extended probability space) follows from the Dambis–Dubins–Schwarz theorem.

(c) follows from (a) and (b). $\square$

§ 4. VERIFICATION LEMMA

Consider

$$V^*(\pi) = \inf_\tau \mathsf{E}_\pi G(h,\tau) = \inf_\tau \mathsf{E}_\pi \left[(1-\pi_\tau^h) + a \int_0^\tau \pi_t^h \, dt + b \int_0^\tau h_t \, dt \right].$$

The verification lemma gives a possibility to check that a strategy $(\overline{h}, \overline{\tau})$ is optimal:

$$\overline{V}(\pi) \equiv \mathsf{E}_\pi G(\overline{h}, \overline{\tau}) = V^*(\pi) \quad \text{for all } \pi \in [0,1]$$

LEMMA 5 [the verification lemma]. Suppose that continuous on $[0,1]$ function $\overline{V}(\pi)$ satisfies the following conditions:

(a) $0 \le \overline{V}(\pi) \le 1-\pi$;

(b) for any admissible strategy (h,τ) the process

$$Y_t \equiv \overline{V}(\pi_t^h) + a\int_0^t \pi_s^h\,ds + b\int_0^t h_s\,ds, \qquad t \ge 0,$$

is a submartingale (w.r.t. $(\mathcal{F}_t^{X^h})_{t\ge 0}$, P_π, $\pi \in [0,1]$);

(c) for the strategy $(\overline{h},\overline{\tau})$ the process $(\overline{Y}_{t\wedge\overline{\tau}})_{t\ge 0}$ with $\overline{Y}_t \equiv \overline{V}(\pi_t^{\overline{h}})$ is a martingale (w.r.t. $(\mathcal{F}_t^{X^h})_{t\ge 0}$, P_π, $\pi \in [0,1]$);

(d) $\mathsf{E}_\pi\overline{\tau} < \infty$;

(e) $\overline{V}(\pi_{\overline{\tau}}^{\overline{h}}) = 1 - \pi_{\overline{\tau}}^{\overline{h}}$.

Then $\overline{V}(\pi) = V^*(\pi)$, $\pi \in [0,1]$, and strategy $(\overline{h},\overline{\tau})$ is optimal.

Proof. Suppose that (h, τ) is a strategy and $\mathsf{E}_\pi \tau = \infty$ for some $\pi \in [0, 1]$. Then $\mathsf{E}_\pi G(h, \tau) = \infty$. Indeed, from the representation

$$G(h, \tau) = I(\tau < \theta) + a(\tau - \theta)^+ + b \int_0^\tau h_t \, dt$$

we have

$$\mathsf{E}_\pi G(h, \tau) \geq a \mathsf{E}_\pi (\tau I(\tau > \theta)) - a \mathsf{E}_\pi \theta.$$

Since $\mathsf{E}_\pi \tau = \mathsf{E}_\pi \tau I(\tau > \theta) + \mathsf{E}_\pi \tau I(\tau \leq \theta)$ and $\mathsf{E}_\pi \tau I(\tau \leq \theta) \leq \mathsf{E}_\pi \theta < \infty$, we conclude that $\mathsf{E}_\pi \tau = \infty$ implies $\mathsf{E}_\pi \tau I(\tau > \theta) = \infty$, and therefore $\mathsf{E}_\pi G(h, \tau) = \infty$. But at the same time

$$\inf_{(h,\tau)} \mathsf{E}_\pi G(h, \tau) \leq 1 - \pi \leq 1.$$

So, we can exclude the strategies (h, τ) with $\mathsf{E}_\pi \tau = \infty$.

By (b) the process $(Y_t)_{t\geq 0}$ is a submartingale. Then by the Optional Sampling Theorem (OST)

$$\mathsf{E}_\pi Y_{\tau\wedge t} \geq \mathsf{E}_\pi Y_0 = \overline{V}(\pi).$$

From the Fatou lemma

$$\mathsf{E}_\pi \limsup_{t\to\infty} Y_{\tau\wedge t} \geq \limsup_{t\to\infty} \mathsf{E}_\pi Y_{\tau\wedge t}.$$

So, by submartingale property of Y and assumption $\mathsf{E}_\pi \tau < \infty$ we get

$$\mathsf{E}_\pi Y_\tau \geq \mathsf{E}_\pi \limsup_{t\to\infty} Y_{\tau\wedge t} \geq \limsup_{t\to\infty} \mathsf{E}_\pi Y_{\tau\wedge t} \geq \overline{V}(\pi). \tag{1}$$

By (a), we have $1 - \pi \geq \overline{V}(\pi)$, so by (1)

$$\begin{aligned}
\mathsf{E}_\pi G(h,\tau) &= \mathsf{E}_\pi \left[(1-\pi_\tau) + a\int_0^\tau \pi_t\,dt + b\int_0^\tau h_t\,dt \right] \\
&\geq \mathsf{E}_\pi \left[\overline{V}(\pi_\tau) + a\int_0^\tau \pi_t\,dt + b\int_0^\tau h_t\,dt \right] \\
&= \mathsf{E}_\pi Y_\tau \geq \overline{V}(\pi).
\end{aligned}$$

This implies that

$$V^*(\pi) = \inf_{(h,\tau)} \mathsf{E}_\pi G(h,\tau) \geq \overline{V}(\pi).$$

Now we give a proof of the inequality

$$V^*(\pi) \leq \overline{V}(\pi).$$

Show first that $\overline{V}(\pi) = \mathsf{E}_\pi \overline{Y}_{\overline{\tau}}$. Since $0 \leq \overline{Y}_t \leq 1 + (a+b)t$ and $\mathsf{E}_\pi \overline{\tau} < \infty$, we get by (c) that the process $(\overline{Y}_{t \wedge \overline{\tau}})_{t \geq 0}$ is a uniformly integrable martingale. By OST

$$\mathsf{E}_\pi \overline{Y}_{\overline{\tau}} = \mathsf{E}_\pi \overline{Y}_0 = \overline{V}(\pi).$$

From this property and (e): $\overline{V}(\pi^{\overline{h}}_{\overline{\tau}}) = 1 - \pi^{\overline{h}}_{\overline{\tau}}$, it follows that

$$\overline{V}(\pi) = \mathsf{E}_\pi \overline{Y}_\tau = \mathsf{E}_\pi \left[\overline{V}(\pi^{\overline{h}}_{\overline{\tau}}) + a \int_0^{\overline{\tau}} \pi^{\overline{h}}_s \, ds + b \int_0^{\overline{\tau}} \overline{h}_s \, ds \right] = \mathsf{E}_\pi G(\overline{h}, \overline{\tau}) \geq V^*(\pi)$$

Together with inequality $\overline{V}(\tau) \leq V^*(\pi)$ this yields that

$$\overline{V}(\pi) = V^*(\pi)$$

and that $(\overline{h}, \overline{\tau})$ is an optimal strategy. $\square$

§ **5.** Here we give some hints how to find a function $\overline{V}(\pi)$ which satisfies the conditions of the verification lemma.

From the representation $Y_t = \overline{V}(\pi_t^h) + a\int_0^t \pi_s^h\,ds + b\int_0^t h_s\,ds$ and assumption $\overline{V} \in C^2$ we find that

$$\begin{aligned} dY_t &= d\overline{V}(\pi_t^h) + (a\pi_t^h + bh_t)\,dt \\ &= \Big[\lambda\overline{V}'(\pi_t^h)(1-\pi_t^h) + \frac{1}{2}\overline{V}''(\pi_t^h)\Big(\frac{r}{\sigma}\,\pi_t^h(1-\pi_t^h)\Big)^2 + a\pi_t^h + bh_t\Big]\,dt \\ &\qquad + \overline{V}'(\pi_t^h)\frac{r}{\sigma}\,\pi_t^h(1-\pi_t^h)\sqrt{h_t}\,d\overline{B}_t. \end{aligned} \tag{2}$$

From here we see that the process $(Y_t)_{t\ge 0}$ is a P_π-submartingale for each $\pi \in [0,1]$ if the term in $[\cdot]$ of (2) is nonnegative for *any* h_t, $t \ge 0$. This term is an affine function in h_t. So, the claim of nonnegativity is reduced to the claim of nonnegativity of these expressions for $h_t \equiv 0$ and $h_t \equiv 1$.

So, for all $x \in [0,1]$ for $f(x) = \overline{V}(x)$ we have the following (variational) inequalities:

$$\begin{aligned} \lambda f'(x)(1-x) + ax &\geq 0, \\ \lambda f'(x)(1-x) + \rho x^2(1-x)^2 f''(x) + ax + b &\geq 0, \end{aligned}$$

where $\rho = r^2/(2\sigma^2)$ is the "signal/noise ratio".

§ 6. HOW TO FIND STRATEGIES $(\overline{h}, \overline{\tau})$ SATISFYING THE VERIFICATION LEMMA

First of all let us note that it is reasonable to pick out two extreme cases $h = 0$ and $h = 1$.

FIRST CASE (complete "NONobservability"): we do not make observations ($h \equiv 0$, $dX_t = 0$) and π_t^0 (i.e., π_t^h for $h = 0$) is nothing else than a priori probability p_t which satisfies the equation

$$dp_t = \lambda(1 - p_t)\, dt.$$

SECOND CASE (complete "observability"): we make observations ($h \equiv 1$) and π_t^1 (i.e., π_t^h for $h = 1$) satisfies the stochastic differential equation

$$d\pi_t^1 = \lambda(1 - \pi_t^1)\, dt + \frac{r}{\sigma^2}\pi_t^1(1 - \pi_t^1)(dX_t^1 - r\pi_t^1\, dt).$$

THIRD CASE ("NONobservability / observability").

By common sense, in the case of expensive cost of observations it is reasonable to make observations when a priori probability p_t (that is, an increasing function $p_t = \pi + (1-\pi)(1-e^{-\lambda t})$) reaches a relatively big level (denoted by A).

After this it is reasonable to begin observations of process X ($dX_t \neq 0$) and declare alarm about appearing of a change-point θ if a posteriori probability reaches some high level.

These three cases suggest an idea to search an optimal strategy in the class of strategies (h, τ) defined by two constant levels A and B, $0 \le A \le B \le 1$, whose different values lead to different regimes of observations.

Taking $A \geq 0$, define processes $\pi_t = \pi_t(A)$ and $X_t = X_t(A)$, $t \geq 0$, as solutions of the following system of equations with degenerate coefficients:

$$d\pi_t = \left(\lambda(1-\pi_t) - \left(\frac{r}{\sigma}\right)^2 \pi_t^2 (1-\pi_t)^2 I(\pi_t > A)\right) dt + \frac{r}{\sigma^2}\,\pi_t(1-\pi_t)\,dX_t \tag{3}$$

and

$$dX_t = I(\pi_t > A)\Big[rI(\theta \leq t)\,dt + \sigma\,dB_t\Big] \tag{4}$$

with $\pi_0 = \pi$, $X_0 = 0$.

REGIME I (see Case 1). Here $A > 0$ and $B = A$. It is the case of the "complete nonobservability". This happens when the cost b for observation is too big.

REGIME II (see Case 2). In this case we assume that $A = 0$. It means that we have a case of the "complete observations". We are in Regime II when the cost b for observation is equal to zero.

REGIME III (see Case 3). In this (most interesting) case we assume that

$$0 < A < B \leq 1.$$

From (3) and (4) we see that if $\pi_t < A$ then $\pi_t = \pi_t^0$ $(= p_t)$. If $A < \pi_t < B$ then we make observations $(dX_t = rI(t \geq \theta)\,dt + \sigma\,dB_t)$ and $\pi_t = \pi_t^1$ $(= \mathsf{P}_\pi(\theta \leq t \,|\, \mathcal{F}_t^X))$.

If $\pi_t(A) > A$, then $\pi_t(A) = \pi_t^1$; if $\pi_t(A) < A$, then $\pi_t(A) = p_t$. ⟨The system (3)–(4) has a weak solution.⟩ At time $\tau = \inf\{t \geq 0 \colon \pi_t \geq B\}$ we declare "alarm" about appearing of a change-point.

We expect ourselves to be within Regime III when the cost b of observation is > 0 but not "big".

§ 7. INVESTIGATION OF (PRINCIPAL) REGIME III. STEFAN PROBLEM.

When operating with Regime III, we look for an optimal strategy (h^*, τ^*) in the class of strategies $(\overline{h}, \overline{\tau})$ which have the form

$$\overline{h}_t = I(\pi_t > \overline{A}), \qquad \overline{\tau} = \inf\{t \geq 0 \colon \pi_t \geq \overline{B}\},$$

where π_t in defined in (3):

$$d\pi_t = \left[\lambda(1-\pi_t) - \left(\frac{r}{\sigma}\right)^2 \pi_t^2 (1-\pi_t)^2 I(\pi_t > \overline{A})\right] dt + \frac{r}{\sigma^2}\,\pi_t(1-\pi_t)\,dX_t.$$

By the verification lemma, we see that for the function $f(x) = \overline{V}(x)$ $(= \mathsf{E}_x G(\overline{h}, \overline{\tau}))$, in addition to the conditions

$$\boldsymbol{\lambda f'(x)(1-x) + ax} \geq 0,$$

$$\boldsymbol{\lambda f'(x)(1-x) + \rho x^2(1-x)^2 f''(x) + ax + b} \geq 0 \qquad (\rho = \frac{r^2}{2\sigma^2})$$

the following conditions should hold:

$$\boldsymbol{\lambda f'(x)(1-x) + ax} = 0, \quad x \in (0, \overline{A}), \tag{5}$$

$$\boldsymbol{\lambda f'(x)(1-x) + \rho x^2(1-x)^2 f''(x) + ax + b} = 0, \quad x \in (\overline{A}, \overline{B}). \tag{6}$$

Also the condition

$$f(x) = 1 - x, \qquad x \in [\overline{B}, 1],$$

should hold.

Let's fix constants $\overline{A}$ and $\overline{B}$ and find solutions of equations (5), (6).

These solutions will contain three unknown constants. In addition to unknown big levels $\overline{A}$ and $\overline{B}$, we have five unknown constants. Thus, we need five additional conditions.

These conditions will be found from the observations that for the function *

$$f(x) = \begin{cases} f_1(x), & x \in [0, \overline{A}), \\ f_2(x), & x \in (\overline{A}, \overline{B}), \\ 1 - x, & x \in [\overline{B}, 1] \end{cases}$$

one can check conditions of the verification lemma.

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* The values $f_1(0)$ and $f_2(\overline{A})$ are defined by continuity: $f_1(0) = \lim_{x\downarrow 0} f_1(x)$ and $f_2(\overline{A}) = \lim_{x\downarrow \overline{A}} f_2(x)$.

The requirement for $f(x)$ to be continuous makes natural the conditions of "continuous fit":

$$f_1(\overline{A}) = f_2(\overline{A}), \qquad f_2(\overline{B}) = 1 - \overline{B}, \tag{7}$$

where $f_2(\overline{B}) = \lim_{x \uparrow \overline{B}} f_2(x)$.

The requirement for the function $f(x)$ to be smooth (which we need, in particular, for application of Itô's formula) leads to the conditions of "smooth fit":

$$f_1'(\overline{A}) = f_2'(\overline{A}), \qquad f_2'(\overline{B}) = -1 \quad (= (1 - \overline{B})') \tag{8}$$

and

$$f_1''(\overline{A}) = f_2''(\overline{A}). \tag{9}$$

(The values of the derivatives at points $\overline{A}$ and $\overline{B}$ are defined by continuity.)

The problem of finding a function $f(x)$ which satisfies five conditions (5)–(9):

$$
\begin{aligned}
&\lambda f'(x)(1-x) + ax = 0, \quad x \in (0, \overline{A}),\\
&\lambda f'(x)(1-x) + \rho x^2 (1-x)^2 f''(x) + ax + b = 0, \quad x \in (\overline{A}, \overline{B}),\\
&f_1(\overline{A}) = f_2(\overline{A}), \qquad f_2(\overline{B}) = 1 - \overline{B},\\
&f_1'(\overline{A}) = f_2'(\overline{A}), \qquad f_2'(\overline{B}) = -1 \quad (= (1-\overline{B})'),\\
&f_1''(\overline{A}) = f_2''(\overline{A}).
\end{aligned}
$$

is named

STEFAN PROBLEM

or

problem with moving unknown boundaries.

§ 8. In this section we get the solution of the Stefan problem for Regime III which will be used for finding the optimal strategy (h^*, τ^*).

For simplicity, we shall write A and B instead of $\overline{A}$ and $\overline{B}$, $0 < A < B \leq 1$. In this problem there are several parameters: $\lambda > 0$, $\sigma > 0$, $r \neq 0$, $a > 0$, and $b \geq 0$. Naturally, the decisions about making observations and declaring alarm depend on values of these parameters. Below we shall see that there exists a critical value

$$b^* = \frac{a\lambda\rho}{(a+\lambda)^2}$$

such that

- if $0 \leq b < b^*$, then in case $b = 0$ we have Regime II (complete observability) and in case $b > 0$ we have Regime III (NONobservability/observability);

- if $b \geq b^*$, then we have Regime I (complete NONobservability).

Assuming that $0 < A < B \leq 1$, consider functions $f_1(x)$ and $f_2(x)$ as solutions of the equations

$$\begin{aligned} \lambda f_1(x)(1-x) + ax = 0, &\qquad x \in (0, A), \\ \lambda f_2'(x)(1-x) + \rho f_2''(x) x^2 (1-x)^2 + ax + b = 0, &\qquad x \in (A, B), \end{aligned}$$

respectively. If $g_1(x) = f_1'(x)$ and $g_2(x) = f_2'(x)$, then we find that

$$g_1'(x) = \frac{\lambda g_1(x) - a}{\lambda(1-x)}, \qquad g_2'(x) = \frac{-ax - b - \lambda(1-x) g_2(x)}{\rho x^2 (1-x)^2}.$$

From the second-order smooth-fit condition

$$g_1'(A) = g_2'(A) \qquad (f_1''(A) = f_2''(A))$$

we find the following relationship between $g_1(A)$ and $g_2(A)$:

$$(\lambda g_1(A) - a)\rho A^2(1 - A) = -\lambda(aA + b) - \lambda^2 g_2(A)(1 - A). \tag{10}$$

Using the first-order smooth-fit condition

$$g_1(A) = g_2(A) \qquad (f_1'(A) = f_2'(A)),$$

we find from (10) that

$$g_1(A) = \frac{a\rho A^2(1 - A) - \lambda(aA + b)}{(1 - A)(\lambda^2 + \lambda\rho A^2)}. \tag{11}$$

From the equation

$$\lambda g_1(x)(1-x) + ax = 0, \quad (\text{i.e.,} \quad \lambda f_1'(x)(1-x) + ax = 0)$$

we find that for $x < A$

$$g_1(x) = -\frac{ax}{\lambda(1-x)}. \tag{12}$$

By continuity this yields that

$$g_1(A) = -\frac{aA}{\lambda(1-A)}. \tag{13}$$

From (11) and (13) we see that the level A should be

$$A = \sqrt{\frac{\lambda b}{a\rho}}. \tag{14}$$

From here it is clear that for $A > 0$ the cost must be positive.

If $b \downarrow 0$, then $A \downarrow 0$ and Regime III $\to$ Regime II. In Regime III, when $(A, B) \neq \varnothing$, we have condition $A < 1$. From concavity of function $f(x)$ and property $f(x) = 1 - x$ for $x \geq B$ it follows that

$$g_1(A) > -1.$$

This inequality and (13) imply that

$$A < \frac{\lambda}{a + \lambda}.$$

Combining this with (14), we see that parameters of our problem should be such that

$$\sqrt{\frac{\lambda b}{a\rho}} < \frac{\lambda}{a + \lambda},$$

i.e., b must satisfy the inequality

$$b < b^* = \frac{\lambda a \rho}{(a + \lambda)^2}.$$

Consider now $x \geq A$. For $g_2(x) = f_2'(x)$, from the basic equation $\lambda f_2'(x)(1-x) + \rho f_2''(x)x^2(1-x)^2 + ax + b = 0$ we obtain that

$$\lambda g_2(x)(1-x) + \rho g_2'(x)x^2(1-x)^2 + ax + b = 0. \qquad (15)$$

Up to a multiplicative constant, the solution of the homogeneous equation $\lambda u(x)(1-x) + \rho u'(x)x^2(1-x)^2 = 0$ is

$$\boldsymbol{u(x) = \Big(\frac{1-x}{x}\Big)^{\alpha} e^{\alpha/x}}, \qquad x > 0, \quad \text{with } \alpha = \frac{\lambda}{\rho}.$$

Thus the general solution of (15) is given by the formula

$$\boldsymbol{g_2(x) = K_2(x)u(x) - u(x)\int_A^x \frac{1}{\rho}\,\frac{ay+b}{y^2(1-y)^2}\,\frac{dy}{u(y)}}. \qquad (16)$$

The value of the constant K_2 is found from the smooth-fit condition $g_1(A) = g_2(A)$ and the formula $g_1(A) = -aA/(\lambda(1-A))$ given above:

$$\boldsymbol{K_2 = -\frac{a}{\lambda}\,\frac{A}{1-A}\,\frac{1}{u(A)} = -\frac{a}{\lambda}\Big(\frac{A}{1-A}\Big)^{1+\alpha} e^{-\alpha/A}}.$$

To find B, we note that

$$\lim_{x\uparrow 1} u(x) = 0, \qquad \lim_{x\uparrow A} g_2(x) = -\infty.$$

At point A

$$g_2(A) = g_1(A) = -\frac{a}{\lambda}\frac{A}{1-A} > -1.$$

Since $g_2(x)$ decreases, one can find $B \in (A, 1)$ such that

$$g_2(B) = -1.$$

This formula together with the formula $A = \sqrt{\lambda b/(a\rho)}$ obtained above allow us to find functions $f_1(x)$ (for $0 \le x \le A$) and $f_2(x)$ (for $A \le x \le B$).

$\boldsymbol{f_2(x)}$**:** For $x \geq A$ we have

$$f_2(x) = \int_A^x g_2(y)\,dy + C_2.$$

Using the condition $f_2(B) = 1 - B$, we find that

$$C_2 = (1 - B) - \int_A^B g_2(y)\,dy.$$

Hence, for $A \leq x \leq B$

$$\boxed{\boldsymbol{f_2(x)} = (1-B) + \int_B^x g_2(y)\,dy}$$

where $g_2(y)$ was given in (16).

$f_1(x)$: Function $f_1(x)$ has the form $f_1(x) = \int_A^x g_1(y)\,dy + C_1$, $x < A$, where C_1 is a constant and $g_1(y) = -ay/(\lambda(1-y))$. So,

$$f_1(x) = \frac{a}{\lambda}[x + \log(1-x)] + C_1.$$

Condition $f_1(A) = f_2(A)$ and representations

$$f_2(x) = \int_A^x g_2(y)\,dy + C_2, \qquad x \geq A,$$
$$f_1(x) = \int_A^x g_1(y)\,dy + C_1, \qquad x < A,$$

imply by continuity that $C_1 = C_2$. So,

$$\boxed{f_1(x) = \frac{a}{\lambda}[x + \log(1-x)] + (1-B) - \int_A^B g_2(y)\,dy}.$$

In particular, $f_1(0) = (1-B) - \int_A^B g_2(y)\,dy$.
Thus, we can formulate the following theorem.

THEOREM 1 (Regime III $\to$ Regime II if $b \downarrow 0$). Under assumption $0 < b < b^* = a\lambda\rho/(a+\lambda)^2$ with $\rho = r^2/(2\sigma^2)$ the solution of the Stefan problem with five boundary conditions at the points A and B is given by

$$f(x) = \begin{cases} f_1(x), & x \in [0, A), \\ f_2(x), & x \in [A, B), \\ 1 - x, & x \in [B, 1], \end{cases}$$

where

$$f_1(x) = \frac{a}{\lambda}[x + \log(1-x)] + (1-B) - \int_A^B g_2(y)\,dy,$$
$$f_2(x) = (1-B) + \int_B^x g_2(y)\,dy,$$

$A = \sqrt{\lambda b/(a\rho)}$ and B is a unique root of the equation $g_2(B) = -1$.

(Functions $g_1(y)$ and $g_2(y)$ are defined in (12) and (16).)

REMARK. If $b \downarrow 0$, then $A \downarrow 0$, i.e., Regime III passes to Regime II of the complete observation. In this case B is a root of the equation

$$\frac{a}{\rho}\int_0^B \frac{u(B)}{u(y)}\,\frac{dy}{y(1-y)^2} = 1, \quad \text{or} \quad \frac{a}{\rho}\int_0^B e^{\lambda[H(y)-H(b)]/\rho}\frac{dy}{y(1-y)^2} = 1,$$

where $H(y) = \log(y/(1-y)) - 1/y$.

Now suppose that in Regime III we have $b \uparrow b^* = a\lambda\rho/(a+\lambda)^2$. Then $A = \sqrt{\lambda b/(a\rho)} \uparrow A^* = \lambda/(a+\lambda)$. At the point A^*

$$f_1'(A^*) = g_1(A^*) = -1.$$

From here and the proof of Theorem 1 it follows that $B \downarrow B^* = A^*$. It demonstrates that in the case $b \geq b^*$ (big cost for observations) we have Regime I, i.e., the case with no observations at all.

The corresponding **STEFAN PROBLEM**:

To find a function $f(x)$ and a level A such that

$$f(x) = \begin{cases} f_1(x), & x \in [0, A], \qquad \lambda f_1'(x)(1-x) + ax = 0, \quad 0 < x < A, \\ 1 - x, & x \in [A, 1], \qquad f_1(A) = 1 - A, \quad f_1'(A) = -1. \end{cases}$$

The solution is

$$f_1(x) = \frac{a}{\lambda}\left[x + \log(1-x) - \log\frac{a}{a+\lambda}\right]. \tag{17}$$

It is evident that

$$\lim_{x\downarrow 0} f_1'(x) = 0, \quad \lim_{x\uparrow A} f_1'(x) = -1, \quad f_1(x) \le 1 - x, \quad \lim_{x\uparrow A} f_1(x) = 1 - A.$$

Thus, we can formulate the following theorem.

THEOREM 2 (Regime I). If $b \geq b^* = a\lambda\rho/(a+\lambda)^2$, where $\rho = r^2/(2\sigma^2)$, then the solution of the corresponding Stefan problem (Regime I) is $f_1 = f_1(x)$ given by (17):

$$f_1(x) = \frac{a}{\lambda}\left[x + \log(1-x) - \log\frac{a}{a+\lambda}\right],$$

and $A = \lambda/(a+\lambda)$.

As the final step we must prove that

> the obtained solution $f = f(x)$ of the Stefan problem coincides, in fact, with the function $V^*(x)$ of the quickest detection problem.

The proof consists in verification of conditions (a)–(e) of the verification lemma.