A double optimal stopping of marked renewal process

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Plan of the presentation

Formulation

K – the number of fishes in a lake;
$T_1, T_2, \dots, T_n$ – the capture times;
$X_1, X_2, \dots, X_n$ – the weights of fishes;
$N(t)$ – the number of fishes caught by time t;
$M(t)$ – total weight of fishes caught by time t;

$$M(t) = \sum_{i=0}^{N(t)} X_i$$

$Z(t)$ – the payoff for stopping at time t;

Goal:

$$EZ(\tau^*) = \sup_{\tau \in \mathcal{T}} EZ(\tau)$$

The history of the basic problem

- [Starr(1974)]
  - $\{T_i\}_{i=0}^{K}$ – i.i.d random variables $\sim \mathcal{E}(\lambda)$;
  - $Z(t) = N(t) - ct$;
- [Starr and Woodroofe(1974)]
  - $\{T_i\}_{i=0}^{K}$ – i.i.d random variables $\sim F(t)$;
  - $F(t)$ is continuous and has DFR$^{(*)}$ or IFR$^{(**)}$;
  - $Z(t) = N(t) - ct$;
- [Starr et al.(1976)Starr, Wardrop, and Woodroofe]
  - $\{T_i\}_{i=0}^{K}$ – i.i.d random variables $\sim F(t)$;
  - $F(t)$ is continuous and has DFR;
  - $Z(t) = g(N(t)) - c(t)$, where g - concave and c - convex;

DFR$^{(*)}$ – Decreasing Failure Rate (i.e. $d(x) = \frac{f(x)}{\bar{F}(x)}$ decreases)

IFR$^{(**)}$ – Increasing Failure Rate

The history of the basic problem

- [Kramer and Starr(1990)], (see also [Fakhre-Zakeri and Slud(1996), Dalal and Mallows(1988)])
  - $\{(X_i, T_i)\}_{i=0}^{K}$ – i.i.d random variables $\sim F(x,t)$;
  - T_i may be dependent on X_i;
  - $Z(t) = M(t) - c(t)$, where c - convex;
- [Ferguson(1997)]
  - $K \sim G(k)$
  - $\{(X_i, T_i)\}_{i=0}^{K}$ i.i.d random variables $\sim F(x,t)$
  - $Z(t) = M(t) - c(t)$, where c - increasing
- [Karpowicz and Szajowski(2008)], [Karpowicz(2009)]
  - $\{(X_{i,n}, T_{i,n})\}_{n=0}^{\infty}$ r.vs; X and T are independent;
  - $X_{i,n}$ are i.i.d. r.v. having $H_i(x)$;
  - $T_{i,n+1} - T_{i,n} \sim F_i(s)$;
  - $Z(s,t) = w(M_s, s, M_t^s, t)$.
  - $EZ(\tau_1^*, \tau_2^*) = \sup_{\tau_1 \in \mathcal{T}} \sup_{\tau_2 \in \mathcal{T}^{\tau_1}} EZ(\tau_1, \tau_2)$.

Definitions and notations

t_0 – finite horizon

FIRST 2 METHODS

- fishes weights
- counting process
- the capture times
- the type
- which are i-th type
- period between successive captures
- utility function
- cost function

Definitions and notations

t_0 – finite horizon

FIRST 2 METHODS

- fishes weights — $\{X_{i,j}\}_{i\in\{1,2\},\ j=0}^{\infty}$
- counting process — $\overrightarrow{N}(t) = (N_1(t), N_2(t))$
- the capture times — $\{(T_n, \mathfrak{z}_n)\}_{n=0}^{\infty}$,
- the type — where $\mathfrak{z}_n \in \{1, 2\}$
- which are i-th type — $n_{i,0} = 0,\ n_{i,k+1} = \inf\{n > n_{i,k} : \mathfrak{z}_n = i\}$
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Definitions and notations

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FIRST 2 METHODS

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- period between successive captures — $T_{i,k} = T_{n_{i,k}}$
- utility function
- cost function

Definitions and notations

t_0 – finite horizon

FIRST 2 METHODS $\xrightarrow{s}$ THIRD METHOD

• fishes weights	• $\{X_{i,j}\}_{i\in\{1,2\},\, j=0}^{\infty}$
• counting process	• $\overrightarrow{N}(t) = (N_1(t), N_2(t))$
• the capture times	• $\{(T_n, \mathfrak{z}_n)\}_{n=0}^{\infty}$,
• the type	• where $\mathfrak{z}_n \in \{1, 2\}$
• which are i-th type	• $n_{i,0} = 0,\ n_{i,k+1} = \inf\{n > n_{i,k} : \mathfrak{z}_n = i\}$
• period between successive captures	• $T_{i,k} = T_{n_{i,k}}$ $S_{i,n} = T_{i,n} - T_{i,n-1}$
• utility function	• $g_{1,i}(\cdot),\ g_1(\cdot)$
• cost function	• $c_{1,i}(\cdot)$

Definitions and notations

t_0 – finite horizon

	FIRST 2 METHODS	$\xrightarrow{s}$ THIRD METHOD
fishes weights	$\{X_{i,j}\}_{i\in\{1,2\},\ j=0}^{\infty}$	$X_{3,0},\ X_{3,1},\ X_{3,2},\ldots$
counting process	$\overrightarrow{N}(t)=(N_1(t),N_2(t))$	$N_3(t)$
the capture times	$\{(T_n,\mathfrak{z}_n)\}_{n=0}^{\infty}$,	$T_{3,0},\ T_{3,1},\ T_{3,2}\ldots$
the type	where $\mathfrak{z}_n\in\{1,2\}$	
which are i-th type	$n_{i,0}=0,\ n_{i,k+1}=\inf\{n>n_{i,k}:\mathfrak{z}_n=i\}$	
period between successive captures	$T_{i,k}=T_{n_{i,k}}$ $S_{i,n}=T_{i,n}-T_{i,n-1}$	$S_{3,n}=T_{3,n}-T_{3,n-1}$
utility function	$g_{1,i}(\cdot),\ g_1(\cdot)$	$g_2(\cdot)$
cost function	$c_{1,i}(\cdot)$	$c_2(\cdot)$

Assumptions for double stopping problem

Assumptions:

For $i \in \{1,2\}$

1. The utility functions $g_i, g_{i,j} : [0,\infty)^3 \longrightarrow [0, W_i]$ are continuous and bounded by W_i.
2. The cost functions $c_i, c_{i,j} : [0, t_0] \longrightarrow [0, C_i]$ are continuous, bounded by C_i and differentiable.
3. $\{X_{i,j}\}_{i\in\{1,2,3\},\, j=0}^{\infty}$ are i.i.d. random variables with known distribution function $H_i(x)$.
4. $\{S_{i,n}\}_{n=0}^{\infty}$ are i.i.d. random variables for fixed $i \in \{1,2,3\}$ with known, continuous distribution functions $F_i(s)$, such that $F_i(t_0) < 1$.
5. The point processes $N_i(t)$ are independent on the sequence of weights $\{X_{i,n}\}_{n=0}^{\infty}$.

Description of the considered processes

Total weight of fishes caught by time t, if the change of the position took place at the time s:

$$M_t^s = \begin{cases} \sum_{i=1}^{2} \sum_{n=1}^{N_i(s \wedge t)} X_{i,n} + \sum_{n=1}^{N_3((t-s)^+)} X_{3,n}, & \text{for } s \leq t, \\ \sum_{i=1}^{2} \sum_{n=1}^{N_i(t)} X_{i,n} & \text{for } s > t. \end{cases}$$

Notations:

$M_{i,t} = \sum_{n=1}^{N_i(t)} X_{i,n}$, $M_t = \sum_{i=1}^{2} M_{i,t}$, $\overrightarrow{M}_t = (M_{1,t}, M_{2,t})$,
$M_{i,n} := M_{i,T_n}$, $M_{3,n}^s := M_{i,T_{3,n}}^s$

Let us fix:

$T_{3,0} = s$, $X_{3,0} = M_s$

The payoffs

The payoff for stopping at time t, if the change of the techniques took place at time s just after the catching by method i is

Payoff when change is on ith method

$$W_i(s,t) = \mathbb{I}_{\{t<s\le t_0\}} w_1(\overrightarrow{M}_t, i, t) + \mathbb{I}_{\{s\le t\le t_0\}} w_2(\overrightarrow{M}_s, i, s, M_t^s, t) - \mathbb{I}_{\{t_0<t\}} C$$

where

$$\begin{aligned} w_1(\vec{m}, i, t) &= g_1(\vec{m}, i, t) - c_1(t), \\ w_2(\vec{m}, i, s, \tilde{m}, t) &= w_1(\vec{m}, i, s) + g_{2,i}(\vec{m}, s, \tilde{m}, t) - c_2(t-s), \\ C &= C_1 + C_2. \end{aligned}$$

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where

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For the global optimization problem, the closer to those problems have been formulated and solved by [Karpowicz(2009)]

$$Z(s,t) = W_{\mathfrak{z}_{N(s)}} = \begin{cases} w_1(\overrightarrow{M}_t, \mathfrak{z}_{N(t)}, t) - c_1(t) & \text{if } t < s \leq t_0, \\ w_2(\overrightarrow{M}_s, \mathfrak{z}_{N(s)}, s, M_t^s, t) & \text{if } s \leq t \leq t_0, \\ -C & \text{if } t_0 < t, \end{cases}$$

Information of the decision maker and his strategies

Definition

$$\begin{aligned}\mathcal{F}_t &= \mathcal{F}_t^{\{1,2\}} = \sigma(X_0, T_0, \mathfrak{z}_0, X_1, T_1, \mathfrak{z}_1, \ldots, X_{N(t)}, T_{N(t)}, \mathfrak{z}_{N(t)});\\ \mathcal{F}_{s,t} &= \sigma(\mathcal{F}_s^{\{1,2\}}, X_{3,0}, T_{3,0}, \ldots, X_{3,N_3((t-s)^+)}, T_{3,N_3((t-s)^+)});\end{aligned}$$

Extra notations:

$\mathcal{F}_{i,n} := \mathcal{F}_{T_{i,n}}$, $\mathcal{F}_n := \mathcal{F}_{T_n}$, $\mathcal{F}_n^s = \mathcal{F}_{s,T_{3,n}}$ and $\mathcal{F}_{s,s} = \mathcal{F}_s$

Information of the decision maker and his strategies

Definition

$$
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\end{aligned}
$$

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Definition

$\mathcal{M}(\mathcal{F}_n)$ ($\mathcal{M}(\mathcal{F}_{i,n})$)– the set of nonegative and $\mathcal{F}_n$ ($\mathcal{F}_{i,n}$)-measurable random variables.

Information of the decision maker and his strategies

Definition

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\begin{aligned}
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\mathcal{F}_{s,t} &= \sigma(\mathcal{F}_s^{\{1,2\}}, X_{3,0}, T_{3,0}, \ldots, X_{3,N_3((t-s)^+)}, T_{3,N_3((t-s)^+)});
\end{aligned}
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Definition

$\mathcal{T}$ – the set of stopping times with respect to the σ-field $\mathcal{F}_t$;
$\mathcal{T}^s$ – the set of stopping times with respect to the σ-field $\mathcal{F}_{s,t}$;

Strategies and goals

Definition

For $i \in \{1,2\}$, $i \neq j$, $n \in \mathbb{N}$ and $n < K$ define:

$$\begin{aligned}
\mathcal{T}_{n,K} &= \{\tau \in \mathcal{T} : \tau \geq 0,\ T_n \leq \tau \leq T_K\}, \\
\mathcal{T}_{i,n,K} &= \{\tau \in \mathcal{T} : \tau \geq 0,\ T_{i,n} \leq \tau \leq T_K\}; \\
\mathcal{T}^s_{n,K} &= \{\tau \in \mathcal{T}^s : \tau \geq s,\ T_{3,n} \leq \tau \leq T_{3,K}\}; \\
\mathcal{T}_i &= \{\tau \in \mathcal{T} : T_{j,N_j(\tau)} \leq T_{i,N_i(\tau)} \leq \tau \leq T_{i,N_i(\tau)+1} \wedge T_{j,N_j(\tau)+1}\}.
\end{aligned}$$

Goal-the global approach

Find two optimal stopping times τ_1^* and τ_2^* in order to maximize the payoff:

$$\mathbf{E}Z(\tau_1^*,\tau_2^*) = \sup_{\tau_1 \in \mathcal{T}} \sup_{\tau_2 \in \mathcal{T}^{\tau_1}} \mathbf{E}Z(\tau_1,\tau_2),$$

where $\tau_1^* < \tau_2^* \leq t_0$

τ_1^* – the moment of stopping the separate methods;

τ_2^* – the moment of stopping the fishing.

The double stopping problem

Optimization violated by technique chosen

Find two optimal stopping times $\tau_1^* \in \mathcal{T}_i$ and $\tau_2^* \in \mathcal{T}^{\tau_1^*}$ in order to maximize the payoff:
$\mathbf{E} W_i(\tau_1^*, \tau_2^*) = \sup_{\tau_1 \in \mathcal{T}_i} \sup_{\tau_2 \in \mathcal{T}^{\tau_1}} \mathbf{E} W_i(\tau_1, \tau_2)$.

Sequential construction of the value

$$
\begin{aligned}
\mathbf{E} W_i(\tau_1^*, \tau_2^*) &= \sup_{\tau_1 \in \mathcal{T}_i} \mathbf{E} W_i(\tau_1, \tau_2^*) = \sup_{\tau_1 \in \mathcal{T}_i} \mathbf{E}\{\mathbf{E}\left[W_i(\tau_1, \tau_2^*) | \mathcal{F}_{\tau_1}\right]\} \\
&= \sup_{\tau_1 \in \mathcal{T}_i} \mathbf{E} \operatorname*{ess\,sup}_{\tau_2 \in \mathcal{T}^{\tau_1}} \mathbf{E}\left[W_i(\tau_1, \tau_2) | \mathcal{F}_{\tau_1}\right] = \sup_{\tau_1 \in \mathcal{T}} \mathbf{E} J_i(\tau_1),
\end{aligned}
$$

where $J_i(s) = \mathbf{E}\{W_i(s, \tau_2^*) | \mathcal{F}_s\} = \operatorname{ess\,sup}_{\tau_2 \in \mathcal{T}^s} \mathbf{E}\{W_i(s, \tau_2) | \mathcal{F}_s\}$.

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Optimization violated by technique chosen

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Sequential construction of the value

$$\begin{aligned} \mathbf{E} W_i(\tau_1^*, \tau_2^*) &= \sup_{\tau_1 \in \mathcal{T}_i} \mathbf{E} W_i(\tau_1, \tau_2^*) = \sup_{\tau_1 \in \mathcal{T}_i} \mathbf{E}\{\mathbf{E}\left[W_i(\tau_1, \tau_2^*) | \mathcal{F}_{\tau_1}\right]\} \\ &= \sup_{\tau_1 \in \mathcal{T}_i} \mathbf{E} \operatorname*{ess\,sup}_{\tau_2 \in \mathcal{T}^{\tau_1}} \mathbf{E}\left[W_i(\tau_1, \tau_2) | \mathcal{F}_{\tau_1}\right] = \sup_{\tau_1 \in \mathcal{T}} \mathbf{E} J_i(\tau_1), \end{aligned}$$

where $J_i(s) = \mathbf{E}\{W_i(s, \tau_2^*) | \mathcal{F}_s\} = \operatorname{ess\,sup}_{\tau_2 \in \mathcal{T}^s} \mathbf{E}\{W_i(s, \tau_2) | \mathcal{F}_s\}$.

Construction of the solution:

1. Calculate $J_i(s)$ and construct the stopping time τ_2^*;
2. Calculate $\mathbf{E} W_i(\tau_1^*, \tau_2^*)$ and construct τ_1^*.

Two anglers optimization problem

Players' payoffs–fixed moments

$$W_{i,j}(s,t) = \mathbb{I}_{\{t<s\le t_0\}} g_{1,i}(\overrightarrow{M}_t, j, t) \quad (1)$$

$$+\mathbb{I}_{\{s\le t\le t_0\}} w_2(\vec{M}_s, j, s, M_t^s, t) - \mathbb{I}_{\{t_0<t\}} C. \quad (2)$$

where $g_{1,i}()$ is the part of ith player payoff based on the first action of the players and $w_2()$ is the component of the final part of the decision process.

Players' payoffs–random moments

Let τ_i, $i = 1,2$ are the strategies of the players to stop individual search and switch to the common search, which is stopped at moment σ. The payoffs of the players are

$$\psi_i(\tau_1,\tau_2) = W_{i,\mathfrak{z}_{N(\tau_1\wedge\tau_2)}}(\tau_1\wedge\tau_2, \sigma^{\tau_1\wedge\tau_2}). \quad (3)$$

Two anglers optimization problem

The construction of the solution:

1. Calculate σ^* and $J_i(s) = \mathbf{E}[g_{1,i}(\overrightarrow{M}_s, \mathfrak{z}_{N(s)}, s) + \mathbb{I}_{\{s \leq \sigma^* \leq t_0\}} w_2(\vec{M}_s, \mathfrak{z}_{N(s)}, s, M^s_{\sigma^*}, \sigma^*) - \mathbb{I}_{\{t_0 < \sigma^*\}} C | \mathcal{F}_s]$;
2. Calculate $(\tau^*_{1,1}, \tau^*_{1,2})$ and $(\mathbf{E}\psi_1(\tau^*_1, \tau^*_2), \mathbf{E}\psi_2(\tau^*_1, \tau^*_2))$ such that $\mathbf{E}\psi_i(\tau^\star_i, \tau^\star_{-i}) \geq \mathbf{E}\psi_i(\tau_i, \tau^\star_{-i})$. for $i \in \{1, 2\}$.

Lemma

[Brémaud(1981)] If $\tau \in \mathcal{T}_{i,n,K}$, then there exists a positive, $\mathcal{F}_{i,n}$-measurable, random variable $R_{i,n}$ such that

$$\tau \wedge T_{j,N_j(T_{i,n})+1} \wedge T_{i,n+1} = (T_{i,n} + R_{i,n}) \wedge T_{j,N_j(T_{i,n})+1} \wedge T_{i,n+1}, \quad a.s., \tag{4}$$

where $R_{i,n}$ is $\mathcal{F}_{i,n} = \mathcal{F}_{T_{i,n}}$-measurable.

Second stopping time, K - fixed

K – the number of fishes in a lake;
s – the moment of changing place;
$m = M_s$ – total weight of fishes caught by time s;

Goal

Find optimal stopping time $\tau^*_{2,K} \in \mathcal{T}^s_{0,K}$ such that:

$$\mathbf{E}\{Z(s, \tau^*_{2,K})|\mathcal{F}_s\} = \operatorname*{ess\,sup}_{\tau_{2,K} \in \mathcal{T}^s_{0,K}} \mathbf{E}\{Z(s, \tau_{2,K})|\mathcal{F}_s\}.$$

Definition

For $n = K, \ldots, 1, 0$

$$\Gamma^s_{n,K} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}^s_{n,K}} E\{Z(s,\tau)|\mathcal{F}_{s,n}\} = E\{Z(s, \tau^*_{2,n,K})|\mathcal{F}_{s,n}\}.$$

Second stopping time

Theorem

Let $s \geq 0$ be the moment of changing place, then:

$$
\begin{aligned}
\Gamma^s_{K,K} &= Z(s, T_{3,K}),\\
\Gamma^s_{n,K} &= \mathbb{I}_{\{T_{3,n}\leq t_0\}} \operatorname*{ess\,sup}_{R_{3,n}\in\mathcal{M}(\mathcal{F}_{s,n})} \Big\{ E\left[\mathbb{I}_{\{S_{3,n+1}\leq R_{3,n}\}}\Gamma^s_{n+1,K}|\mathcal{F}_{s,n}\right]\\
&+ \bar{F}_3(R_{3,n})[\mathbb{I}_{\{R_{3,n}\leq t_0-T_{3,n}\}} w(M_s, s, M^s_n, T_{3,n}+R_{3,n})\\
&- C\mathbb{I}_{\{R_{3,n}>t_0-T_{3,n}\}}]\Big\} - C\mathbb{I}_{\{T_{3,n}>t_0\}} \ a.s.
\end{aligned}
$$

Second stopping time, K - fixed

Lemma

$$\Gamma^s_{n,K} = \gamma^{s,\mathfrak{z}_{N(s)},M_s}_{K-n}(M^s_n, T_{2,n}) \quad n = K, \ldots, 0,$$

where

$$\begin{aligned}\gamma^{s,k,\overrightarrow{m}}_j(\widetilde{m}, t) &= \mathbb{I}_{\{t \le t_0\}}\left\{w_2(\overrightarrow{m}, k, s, \widetilde{m}, t) + y_{2,j}(\overrightarrow{m}, \widetilde{m}, t - s, t_0 - t)\right\} \\ &- C\mathbb{I}_{\{t > t_0\}}\end{aligned}$$

and $y_{2,j}(a, \widetilde{a}, b, t_0 - t)$ is given recursively as follows:

$$\begin{aligned}y_{2,0}(a, \widetilde{a}, b, t_0 - t) &= 0, \\ y_{2,j}(a, \widetilde{a}, b, t_0 - t) &= \max_{0 \le r \le t_0 - t} \phi_{2,y_{2,j-1}}(a, \widetilde{a}, b, t_0 - t, r).\end{aligned}$$

Second stopping time, K - fixed

Lemma c.d.

and the function $\phi_{2,\delta}(a, \widetilde{a}, b, c, r)$ is given by the equation:

$$\begin{aligned}\phi_{2,\delta}(a, \widetilde{a}, b, c, r) &= \int_0^r \bar{F}_3(z)\{\alpha_3(z)[E(g_2(a + X_2) - g_2(a)) \\ &+ E\delta(a + X_2, b + z, c - z)] - c_2'(b + z)\}dz.\end{aligned}$$

where $\alpha_i = \frac{f_i}{\bar{F}_i}$, $\Delta_i(\hat{a}) = \mathbf{E}[g_i(\hat{a} + X_i) - g_i(\hat{a})]$.

Second stopping time, K - fixed

Definition

$B = B([0,\infty) \times [0,t_0] \times [0,t_0])$ − the space of all bounded continuous functions with the norm $\|\delta\| = \sup_{a,b,c} |\delta(a,b,c)|$.

Remark

B with the norm supremum is complete space.

Definition

The operator $\Phi_2 : B \to B$ is given by

$$(\Phi_2\delta)(a,b,c) = \max_{0 \le r \le c} \phi_{2,\delta}(a,b,c,r).$$

Second stopping time, K - fixed

Remark

$$y_{2,j}(a,b,c) = (\Phi_2 y_{2,j-1})(a,b,c);$$

Lemma

There exists function $r^*_{2,j}(a,b,c)$ such that:

$$y_{2,j}(a,b,c) = \phi_{2,y_{2,j-1}}(a,b,c,r^*_{2,j}(a,b,c)).$$

Corollary

The function $\gamma_j^{s,m}(\widetilde{m},t)$ takes the maximum value for

$$r = r^*_{2,j}(\widetilde{m}-m, t-s, t_0-t).$$

Second stopping time, K - fixed

Theorem

If

$$R^*_{3,i} = r^*_{3,K-i}(M^s_i - M_s, T_{2,i} - s, t_0 - T_{2,i}),$$
$$\eta^s_{n,K} = K \wedge \inf\{i \geq n : R^*_{2,i} < S_{2,i+1}\},$$

then the stopping time $\tau^*_{2,n,K} = T_{2,\eta^s_{n,K}} + R^*_{2,\eta^s_{n,K}}$ is optimal in the class $\mathcal{T}^s_{n,K}$ and

$$\Gamma^s_{n,K} = E\left[Z(s,\tau^*_{2,n,K})|\mathcal{F}_{s,n}\right].$$

Second stopping time, K - fixed

Algorithm:

If you have n fishes at the time $t_{2,n} = T_{2,n}$ which weight $m_n^s = M_n^s$ then:

1. Calculate

$$r_{2,n}^* = r_{2,K-n}^*(m_n^s - m, t_{2,n} - s, t_0 - t_{2,n});$$

2. Wait by the time $t_{2,n} + r_{2,n}^*$;
   - If the next capture occurs before the time $t_{2,n} + r_{2,n}^*$ then calculate

$$r_{2,n+1}^* = r_{2,K-(n+1)}^*(m_{n+1}^s - m, t_{2,n+1} - s, t_0 - t_{2,n+1})$$

     and repeat the procedure;
   - Else $\longrightarrow$ STOP

Second stopping time, $K \longrightarrow \infty$

Let us assume that $K \longrightarrow \infty$.

Goal:

Find stopping time $\tau_2^* \in \mathcal{T}^s$, which is optimal in the class $\mathcal{T}^s$:

$$J(s) = E\{Z(s,\tau_2^*)|\mathcal{F}_s\} = \operatorname*{ess\,sup}_{\tau_2 \in \mathcal{T}^s} E\{Z(s,\tau_2)|\mathcal{F}_s\}.$$

Lemma

If $F_2(t_0) < 1$ then the operator $\Phi_2 : B \to B$ is a contraction.

Second stopping time, $K \longrightarrow \infty$

Lemma

There exists $y_2 \in B$ such that

$$y_2 = \Phi_2 y_2$$

and the function $y_2 \in B$ is the limit of the sequence $y_{2,K}$, when K tends to infinity.

Proof:

- $y_{2,K} \in B$ and B is complete space,
- The operator Φ_2 is a contraction,
- Banach Fixed Point Theorem

Second stopping time, $K \longrightarrow \infty$

Lemma

The limit $\gamma^{s,m} = \lim_{K\to\infty} \gamma_K^{s,m}$ exists and

$$\begin{aligned}\gamma^{s,m}(\widetilde{m}, t) &= \mathbb{I}_{\{t\le t_0\}}\left[w_2(m, s, \widetilde{m}, t) + y_2(\widetilde{m} - m, t - s, t_0 - t)\right] \\ &- C\mathbb{I}_{\{t>t_0\}}.\end{aligned}$$

Remark

$$\gamma^{s,m}(m, s) = \mathbb{I}_{\{s\le t_0\}} u(m, s) - C\mathbb{I}_{\{s>t_0\}},$$

where

$$u(m, s) = g_1(m) - c_1(s) + g_2(0) - c_2(0) + y_2(0, 0, t_0 - s).$$

Second stopping time, $K \longrightarrow \infty$

Lemma

The function $\bar{y}(t_0 - s) = y(0, 0, t_0 - s)$ has bounded left-hand sided derivative with respect to s for $s \in (0, t_0]$.

Proof:

- The operator Φ_2 is contraction and $y_2 = \Phi_2 y_2$;
- Taylor's Formula;

Lemma

The function $u(m, s)$ is continuous, bounded and measurable with bounded left-hand sided derivatives with respect to s.

Second stopping time, $K \longrightarrow \infty$

Theorem

If $F_2(t_0) < 1$, then

- The limit $\tau^*_{2,n} = \lim_{K\to\infty} \tau^*_{2,n,K}$ *a.s.* exists.
- The stopping time $\tau^*_{2,n} \leq t_0$ is an optimal stopping rule in the set $\mathcal{T}^s \cap \{\tau \geq T_{2,n}\}$.
- $E\{Z(s,\tau^*_{2,n})|\mathcal{F}_{s,n}\} = \gamma^{s,m}(M^s_n, T_{2,n}) \quad a.s.$

Corollary

$$\begin{aligned} J(s) &= E\left[Z(s,\tau^*_2)|\mathcal{F}_s\right] = \gamma^{s,M_s}(M_s,s) \\ &= \mathbb{I}_{\{s\leq t_0\}} u(M_s,s) - C\mathbb{I}_{\{s>t_0\}} \;\; a.s. \end{aligned}$$

Second stopping time, $K \longrightarrow \infty$

Proof:

- The sequence $\tau^*_{2,n,K}$ is nondecreasing with respect to K and bounded by t_0;
- $V(t) = t - T_{2,N_2(t)} \Rightarrow \xi^s(t) = (t, M^s_t, V(t))$ is Markov process;
- $Z(s,t) = p^{s,m}(\xi^s(t))$;
- $\mathcal{A}p^{s,m}(t,\widetilde{m},v) = \frac{f_2(v)}{\bar{F}_2(v)}[Eg_2(\widetilde{m}+X_2-m) - g_2(\widetilde{m}-m)] - c_2'(t-s)$;
- $p^{s,m}(\xi^s(t)) - p^{s,m}(\xi^s(s)) - \int_s^t (\mathcal{A}p^{s,m})(\xi^s(z))dz$ is a martingale;
- From Dynkin formula and dominated convergence Theorem:

$$E\left[Z(s,\tau^*_{2,n})|\mathcal{F}_{s,n}\right] = \lim_{K\to\infty} E\left[Z(s,\tau^*_{2,n,K})|\mathcal{F}_{s,n}\right]$$
$$= \lim_{K\to\infty} \gamma^{s,M_s}_{K-n}(M^s_n, T_{2,n}) = \gamma^{s,M_s}(M^s_n, T_{2,n}) \ a.s.$$

- $E\left[Z(s,\tau)|\mathcal{F}_{s,n}\right] \le E\left[Z(s,\tau^*_{2,n})|\mathcal{F}_{s,n}\right] \quad \forall \tau \in \mathcal{T}^s \cap \{\tau_{2,n} \ge T_{2,n}\}$

First stopping time

Corollary

- $J(s) = \mathbb{I}_{\{s \leq t_0\}} u(M_s, s) - C\mathbb{I}_{\{s > t_0\}}$;
- The function $u(m, s)$ is continuous, bounded, measurable with bounden left-hand sided derivatives with respect to s;

$\Longrightarrow$ $J(s)$ has similar structure like the process $Z(s, t)$ and the rest of the calculations runs like for second stopping time.

First stopping time

Theorem

If $F_1(t_0) < 1$ then

- The limit $\tau^*_{1,n} = \lim_{K\to\infty} \tau^*_{1,n,K}$ *a.s.* exists;
- The stopping time $\tau^*_{1,n} \leq t_0$ is an optimal stopping rule in the set $\mathcal{T} \cap \{\tau \geq T_{1,n}\}$;
- $E\left[J(\tau^*_{1,n})|\mathcal{F}_n\right] = \gamma(M_n, T_{1,n}) \quad a.s.$

Optimal revenue

$$EZ(\tau^*_1, \tau^*_2) = EJ(\tau^*_1) = \gamma(M_0, T_{1,0}) = \gamma(0, 0),$$

where $\tau^*_1 = \tau^*_{1,0}$ and $\tau^*_2 = \tau^*_{2,0}$ were calculated above.

Nash value and point

Let us denote $\Gamma_{i,j,n,K} = \mathbf{E}\psi_i(\tau_1^*, \tau_2^*)$, when $\tau_1^*, \tau_2^* \in \mathcal{T}_{j,n,K}$.

Theorem

If $F_i(t_0) < 1$, $i \in \{1, 2\}$ then

- The limit $\tau_{i,j,n}^* = \lim_{K\to\infty} \tau_{i,j,n,K}^*$ *a.s.* exists;
- The stopping times $\tau_{i,j,n}^* \leq t_0$, $\imath \in \{1, 2\}$ form a Nash point in the set $\mathcal{T} \cap \{\tau \geq T_{j,n}\}$;
- $\mathbf{E}\left[J_i(\tau_{1,n}^* \wedge \tau_{2,n}^*)|\mathcal{F}_{j,n}\right] = \gamma_{i,j}(M_n, T_{j,n}) \quad a.s.$

Nash value

$$\mathbf{E}\psi_i(\tau_1^* \wedge \tau_2^*) = \mathbf{E}J_i(\tau_1^* \wedge \tau_2^*) = \gamma_{i,1}(M_0, T_{1,0}) = \gamma_{i,1}(0, 0),$$

where $\tau_1^* = \tau_{1,1,0}^*$ and $\tau_2^* = \tau_{2,1,0}^*$ were calculated above.

Nash value and point

Lemma

$\Gamma_{i,j,n,K} = \gamma_{i,K-n}(M_n, j, T_{j,n})$ for $n = K, \ldots, 0$, where the sequence of functions $\gamma_{i,j}$ can be expressed as:

$$\gamma_{i,k}(\overrightarrow{m}, j, s) = \mathbb{I}_{\{s \leq t_0\}} \left\{ u(m, j, s) + y_{i,j}(\overrightarrow{m}, k, s, t_0 - s) \right\} - C\mathbb{I}_{\{s > t_0\}}$$

and $y_{i,j}(\vec{a}, k, b, c)$ is given recursively as follows: $\vec{y}_0(a, k, b, c) = 0$,

$$\vec{y}_j(a, k, b, c) = val\vec{\phi}_{\vec{y}_{j-1}}(a, b, c, r, s),$$

where, for $i \neq j$, $i, j \in \{1, 2\}$

$$\begin{aligned} \phi_{i,\delta}(a, b, c, r_i, r_j) = & \int_0^{r_i} \bar{F}_i(z) F_j(b + z - r_2) \{\alpha_i(z) \big[\Delta_i(a) \\ & + E\delta(a + X_i, b + z, c - z)\big] \\ & - (\bar{y}'_{-}(c - z) + c'_1(b + z))\} dz. \end{aligned}$$

Infinitesimal operator

Notation

$$\begin{aligned}\zeta_2(t) &= \mathcal{A}p^{s,m}(\xi^s(t))\\ &= \frac{f_2(V_2(t))}{\bar{F}_2(V_2(t))}[Eg_2(M_t^s + X_2 - m) - g_2(M_t^s - m)]\\ &- c_2'(t-s),\end{aligned}$$

Notation

$$\begin{aligned}\zeta_1(s) &= \mathcal{A}p(\xi(s))\\ &= \frac{f_1(V_1(s))}{\bar{F}_1(V_1(s))}[Eg_1(M_t + X_1) - g_1(M_t)]\\ &- \left[\bar{y}_{2-}'(t_0 - s) + c_1'(s)\right].\end{aligned}$$

Monotone case

Remark

- If the process $\zeta_i(t)$, $i \in \{1,2\}$, has decreasing paths, then the optimal stopping time is given by:

$$\tau^*_{i,n} = \inf\{t \in [T_{i,n}, t_0] : \zeta_i(t) \le 0\}$$

- If the process $\zeta_i(t)$ has nondecreasing paths, then the optimal stopping time is given by: $\tau^*_{i,n} = t_0$ for all $n \in \mathbb{N}$.

Example 1

If for $i \in \{1, 2\}$

- S_i has exponential distribution with constant rate λ_i;
- c_i is convex;
- g_i is increasing and concave;
- s-the moment of changing place, $m = M_s$;
- $t_{2,n} = T_{2,n}$, $m_n^s = M_n^s$;
- $t_{1,n} = T_{1,n}$, $m_n = M_n$

Solution:

$$\tau_{2,n}^* = \inf\{t \in [t_{2,n}, t_0] : \lambda_2[Eg_2(m_n^s + X_2 - m) - g_2(m_n^s - m)] \le c_2'(t - s)\}$$

$$\tau_{1,n}^* = \inf\{t \in [t_{1,n}, t_0] : \lambda_1 [Eg_1(m_n + X_1) - g_1(m_n)] \le c_1'(t)\}$$

Example 2

If for $i \in \{1, 2\}$

- S_i has exponential distribution with constant rate λ_i;
- c_i is concave;
- g_i is increasing and convex;

Solution:

$$\tau^*_{1,n} = \tau^*_{2,n} = t_0.$$

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