An optimal arrival time problem for queuing system

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Queuing system $?/M/1/0$

Players try to send their requests in the system:

- discipline of request arrivals is unknown;

- request service time is exponentialy distributed with parameter $1/\mu$ ($\mu > 0$);

- only one request can be served at one time;

- system with loss: arriving request is failed if another request is served in the system already.

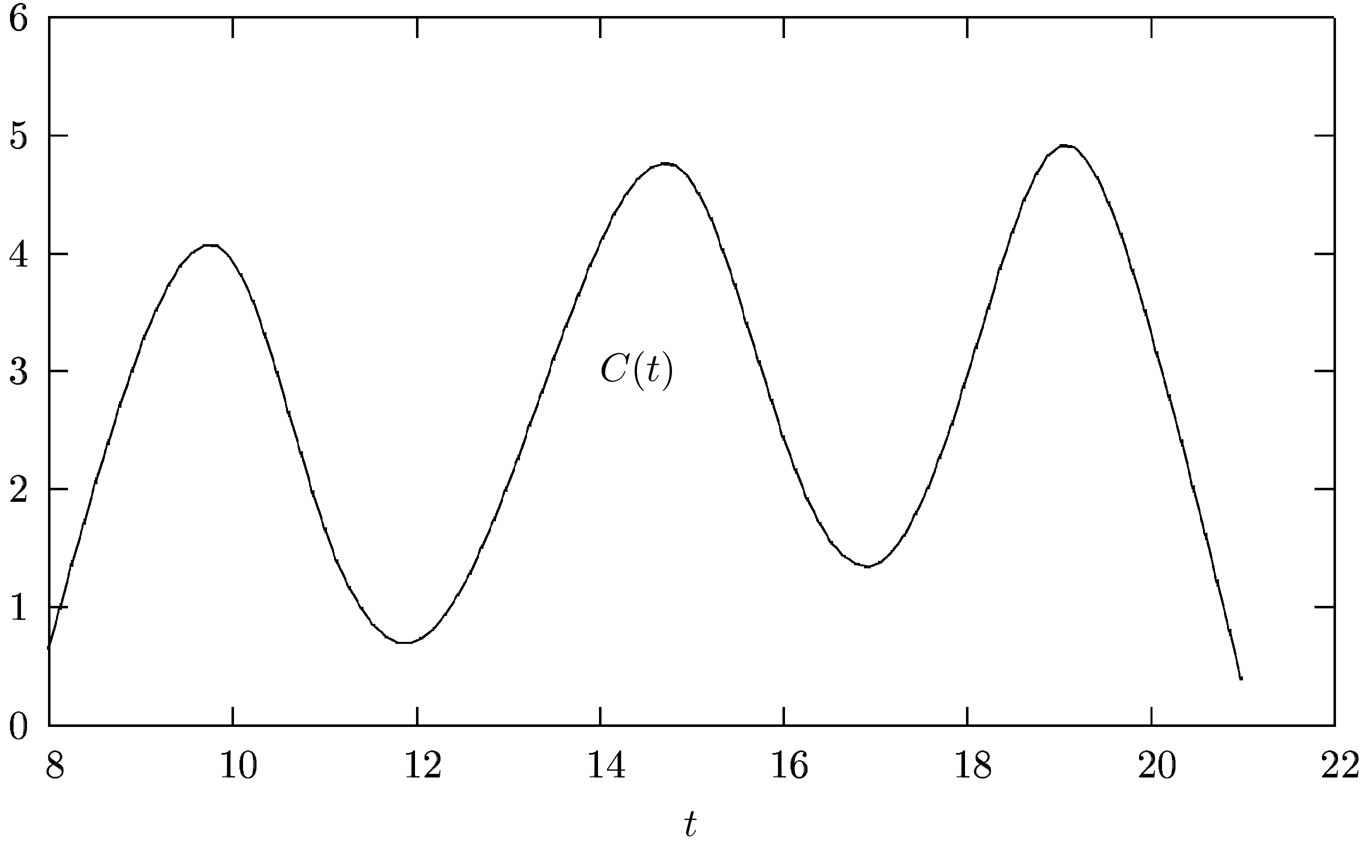
Examples: transport, hotel, etc. reservation systems; filesystem.

Payoffs

"Convinience" function $C(t)$ – a desirability to start request service at the moment t.

• In case the request arrives at time t and is served successfully, player obtains $C(t)$, otherwise he obtains 0. $C(t) > 0$ – profit, $C(t) < 0$ – costs.

• Suppose that player needs to try the system in any case even if he incur costs.



Example: Convinience to access the Internet in the office from one computer during working day.

Strategies and payoffs

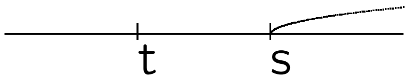
Player's mixed strategy is a **distribution density** of arrival time moments t on **time interval** $[t_0, T]$.

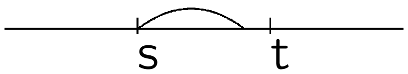
Payoff function is the expected player's profit (or costs) at the moment t.

An **equilibrium** players' strategies must maximize their payoff functions on the interval $[t_0, T]$.

The case of two players

Two players choose arrival time moments: t and s. By symmetry their optimal behavior will be the same. Probability of request service starting at the moment t for player when his opponent uses mixed strategy $g(\cdot)$ is:





$$\int_t^{\infty} g(s)ds + \int_{-\infty}^{t} g(s)(1 - e^{-\mu(t-s)})ds,$$

and his payoff function is as follows:

$$H(t,g) = C(t)\left(\int_{-\infty}^{t} g(s)(1 - e^{-\mu(t-s)})ds + \int_t^{\infty} g(s)ds\right)$$
$$= C(t)\left(1 - \int_{-\infty}^{t} g(s)e^{-\mu(t-s)}ds\right).$$

The equilibrium g, t_0 and T must satisfy:

1. for all $t \in [t_0, T]$ $\quad \frac{\partial H(t,g)}{\partial t} = 0$;

2. $\int\limits_{t_0}^{T} g(t)dt = 1$;

3. for all $t \in (-\infty, \infty)$ $\quad g(t) \geq 0$ and $H(t, g) \leq H(t_0, g)$.

Transforming $\frac{\partial H(t,g)}{\partial t} = 0$ we obtain

$$\int\limits_{-\infty}^{t} g(s)e^{\mu s}ds = \frac{C(t)g(t) - C'(t)}{\mu C(t) - C'(t)}e^{\mu t}, \qquad (1)$$

and then

$$\begin{aligned} &g'(t)\left(C^2(t)\mu - C(t)C'(t)\right) + \\ &g(t)\left(C(t)C''(t) - 2(C'(t))^2 + C'(t)C(t)\mu\right) \\ &-\mu(C(t)C''(t) - 2(C'(t))^2 + C'(t)C(t)\mu) = 0. \end{aligned} \qquad (2)$$

The solution for (2) is $\quad g(t) = Ke^{I(t)} + \mu, \quad$ where

$$I(t) = \int\limits_{t_0}^{t} \frac{\frac{C''(\tau)}{C'(\tau)} - 2\frac{C'(\tau)}{C(\tau)} + \mu}{1 - \mu\frac{C(\tau)}{C'(\tau)}}d\tau.$$

The constant K can be found substituting $g(t)$ in (1) and letting $t = t_0$.

The equilibrium strategy $g(t)$ is

$$g(t) = \begin{cases} \left(\frac{C'(t_0)}{C(t_0)} - \mu\right) e^{I(t)} + \mu, & \text{if } t \in [t_0, T], \\ 0, & \text{otherwise}, \end{cases}$$

where

$$I(t) = \int_{t_0}^{t} \frac{\frac{C''(\tau)}{C'(\tau)} - 2\frac{C'(\tau)}{C(\tau)} + \mu}{1 - \mu\frac{C(\tau)}{C'(\tau)}} d\tau.$$

The equilibrium payoff function is continuous and is as follows:

$$H(t) = \begin{cases} C(t) & \text{for } t \in (-\infty, t_0) \\ C(t_0) & \text{for } t \in [t_0, T] \\ C(t)\left(1 - e^{-\mu(t-T)}\left(1 - \frac{C(t_0)}{C(T)}\right)\right) & \text{for } t \in (T, \infty). \end{cases}$$

At the same time must be

$$\int_{t_0}^{T} g(t)dt = 1 \text{ and } g(t) \geq 0 \text{ for all } t \in (-\infty, \infty),$$

$$C(t) \leq C(t_0) \text{ for } t \in (-\infty, t_0],$$

$$C(t_0) \geq C(t)\left(1 - e^{-\mu(t-T)}\left(1 - \frac{C(t_0)}{C(T)}\right)\right) \text{ for } t \in [T, \infty).$$

An exponential "convinience" function case

Let $C(t) = ae^{bt}$ for $t \geq t_0$ and $C(t) = ae^{bt_0} = C(t_0)$ for $t \leq t_0$. Then

$$g(t) = (b - \mu)e^{-b(t-t_0)} + \mu.$$

Consider t_0 is known, then right bound of $[t_0, T]$ can be found from

$$\mu = \frac{be^{-b(T-t_0)}}{e^{-b(T-t_0)} - 1 + b(T - t_0)}.$$

Then on $[t_0, T]$ the equilibrium payoff will be $H(t, g) \equiv ae^{bt_0}$.
To provide feasible solution following condition must be performed for $t \geq T$

$$ae^{bt_0} \geq ae^{bt}\left(1 - e^{-\mu(t-T)}(1 - e^{-b(t-t_0)})\right).$$

When $b > 0$ and $a < 0$ it is true.

A parabolic "convinience" function case

Let $C(t) = at(1-t)$ with $a > 0$. Then

$$g(t) = \frac{(1 - 2t - \mu t + \mu t^2)t_0(1 - t_0)}{t^2(1-t)^2} + \mu.$$

It is proven that exist t_0 and T:

$$0 < t_0 < \frac{1}{2} \leq T < 1,\ g(T) = 0 \text{ and } \int_{t_0}^{T} g(t)dt = 1$$

and such strategy $g(t)$ on $[t_0, T]$ is a feasible equilibrium.

The case of ≥ 3 players

Consider the queue system that is used by $n+1$ players, wich select arrival times $\tau_1, \ldots, \tau_n$ and t correspondingly.

Suppose that the player uses a pure strategy t when other players with numbers $i = 1, \ldots, n$ use the same mixed strategies $g(\cdot)$. His payoff function is

$$H(t, g^n) = C(t) P_n(t, g),$$

where $P_n(t, g)$ is a probability that his request arriving at time t is served successfully.

$$P_n(t,g) := 1 - n \int\limits_{-\infty}^{t} g(\tau_1) e^{-\mu(t-\tau_1)} (1 - (n-1) \int\limits_{-\infty}^{\tau_1} g(\tau_2) e^{-\mu(\tau_1-\tau_2)} \cdot$$

$$\cdot (1 - (n-2) \int\limits_{-\infty}^{\tau_2} \ldots) d\tau_2) d\tau_1,$$

or recurrently

$$P_1(t,g) = 1 - \int\limits_{-\infty}^{t} g(\tau_1) e^{-\mu(t-\tau_1)} d\tau_1$$

$$P_2(t,g) = 1 - 2 \int\limits_{-\infty}^{t} g(\tau_1) e^{-\mu(t-\tau_1)} P_1(\tau_1, g) d\tau_1$$

$$\ldots$$

$$P_n(t,g) = 1 - n \int\limits_{-\infty}^{t} g(\tau_1) e^{-\mu(t-\tau_1)} P_{n-1}(\tau_1, g) d\tau_1.$$

“Convinience” for uniform strategies

Let $g(t) = 1$ on $t \in [0, 1]$.

$$P_n(t, g) =$$

$$1 + \frac{n!(-1)^n}{\mu^n}\left((1 - e^{-\mu t})\sum_{i=0}^{n-1}\frac{(-\mu)^i}{i!} - e^{-\mu t}\sum_{i=1}^{n-1}\frac{(\mu t)^i}{i!}\sum_{j=0}^{n-i-1}\frac{(-\mu)^j}{j!}\right) =$$

$$= 1 + \frac{n!}{(-\mu)^n}\left(\sum_{i=0}^{n-1}\frac{(-\mu)^i}{i!} - e^{-\mu t}\sum_{i=0}^{n-1}\frac{(-\mu)^i(1-t)^i}{i!}\right).$$

$C_n(t) = C_n(0)/P_n(t, g)$ on $t \in [0, 1]$.

If we consider $n \to \infty$ then $P_n(t, g) \to 1$ and $C_n(t)$ tends to constant not depending on t.

“Convinience” for exponential strategies

Let $g(t) = \lambda e^{-\lambda t}$, $t \geq 0$.

$$P_n(t,g) = 1 + \sum_{i=1}^{n} \frac{n!}{(n-i)!} \frac{e^{-i\lambda t} - e^{-(n-i)\lambda t - \mu t}}{\prod_{j=1}^{i} (j - \mu/\lambda)}.$$

$C_n(t) = C_n(0)/P_n(t,g)$ on $t \geq 0$.

If μ is near 0, i.e. average service time is large, and $t < \infty$ then

$$\prod_{j=1}^{i} (j - \mu/\lambda) \approx i! \text{ , } e^{-\mu t} \approx 1 \text{ and } P_n(t,g) \approx e^{-n\lambda t}.$$

Then, if $n \to \infty$, $P_n(t,g) \to 0$, and $C_n(t) \to \infty$.